THE INSTABILITY OF PARTIALLY RESTRAINED COLUMNS AND PLATES

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THE INSTABILITY OF PARTIALLY RESTRAINED

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GLOSSARY OF SYMBOLS

a, b	dimensions of rectangular plate
D	flexural rigidity of plate
EI	flexural rigidity of column
K, K_1, K_2, K_3, K_4	rotational spring constant
K_ℓ	lateral spring constant
ℓ	length of column
N	uniform axial compressive force along the edge of plate
P	axial load on column
$P_{\beta_1\beta_2}$	axial load, P, corresponding to restraint parameters β_1, β_2
Q	lateral concentrated force
U	total potential energy
w	lateral deflection
x, y	cartesian coordinates
α	nondimensional buckling load parameter, $\alpha = \dfrac{P\ell^2}{\pi^2 EI}$
β	rotational restraint parameter for column, $\left(\beta = \dfrac{K\ell}{EI}\right)$ and for plate, $\left(\beta = \dfrac{Kb}{D}\right)$
β_i	for column, $\beta_i = \dfrac{K_i \ell}{EI}$, and for plate, $\beta_i = \dfrac{K_i b}{D}$
γ	plate aspect ratio, $\gamma = \dfrac{a}{b}$
ξ, η	nondimensional coordinates, $\xi = \dfrac{x}{\ell}$ for column, $\xi = \dfrac{x}{a}$ and $\eta = \dfrac{y}{b}$ for plate
λ	nondimensional axial load, $\lambda = \sqrt{\dfrac{P\ell^2}{EI}}$

GLOSSARY OF SYMBOLS (Continued)

μ	nondimensional buckling coefficient, $$\mu = \frac{Nb^2}{\pi^2 D}$$
$\mu_{1,2,3,4}$	μ corresponding to the restraint parameters β_1, β_2, β_3, β_4
Ω	lateral restraint coefficient, $\Omega = \frac{K_\ell \cdot \ell^3}{EI}$

Subscripts:

cr	critical value
x, y, ξ, η	differentiation, for example $w_x = \frac{dw}{dx}$

SUMMARY

Instability is one of the main issues in structural design. It is well recognized that the behavior of a structural element in a destabilizing environment is highly dependent upon the restraints at its boundaries. In most cases the determination of the exact solution to such problems is neither practical nor possible. An approximate solution must be used. Rayleigh-Ritz is one of the processes commonly employed to obtain approximate eigenvalues. The choice of a suitable displacement function is the key to the successful use of Rayleigh-Ritz process for determining critical loads of structures. No systematic method for making the appropriate choice for elastically restrained columns and plates existed. In this dissertation, a systematic method of generating the displacement function from those which correspond to the several combinations of zero and infinite restraints is outlined. In brief, a first approximation is made by taking a linear combination of the extreme functions. Second and higher approximations are obtained by using additional terms developed from these functions. The general displacement function is expressed as a power series of product of the extreme functions. For all problems investigated, the linear terms give results well within normal engineering accuracy and when higher order terms are considered the convergence is outstanding.

Using this technique, simple accurate formulae for the instability loads of elastically restrained columns are developed. A

convenient expression for obtaining the instability loads of a rotationally restrained rectangular plate under uniform biaxial compression is devised.

A technique for determining the buckling coefficients for a uniaxially compressed rectangular plate with simply supported loaded edges and rotationally restrained unloaded sides, from the corresponding results of the plates with simply supported loaded edges and simply supported and clamped unloaded sides, has been devised.

Buckling coefficients obtained by Rayleigh-Ritz process are tabulated for rotationally restrained rectangular plates under uniform uniaxial and biaxial compression.

The exact solution for the instability loads of uniaxially and biaxially compressed rectangular plates with rotationally restrained boundaries is presented in Chapter IV. A finite difference formulation for the above cases of plates is outlined in Appendix C. The comparison of the buckling coefficients obtained by Rayleigh-Ritz, finite difference method and exact solution is given in the end.

CHAPTER I

INTRODUCTION

Man's interest in the strength of compression members and his gradually increasing knowledge in this regard is obvious from the structures of antiquity. In ancient times the design knowledge was acquired through a trial and error process but eventually this led to systematic study. In fact as early as 1729, Musschenbroek [1] built a testing machine for the determination of the strength of columns. The first theoretical development appeared in 1757, when Euler [2] presented his classical theory of column stability. Reference [3] broadly reviews the experimental work that followed.

Plate instability became important when civil engineers began to use metal structures for bridges and allied systems. Fairbairn and Hodgkinson [4], in 1845, tested various tubular sections in connection with the design of the Conway and Britannia bridges. Fifty years were to elapse before Bryan [5] investigated the stability of the compressed plate theoretically. Since then many analytical and experimental studies have been made. The experimental research in this area has been surveyed in detail in Reference [6].

It is clear from the reviews of the References [7] and [8] that many studies have been made of plate instability. Generally the solutions depend upon satisfying the governing differential equation and the associated boundary conditions either exactly or as closely as possible. In the former case we obtain the accuracy at the expense of time and labor; in the latter we frequently sacrifice accuracy for ease and speed. This is particularly so when the problem involves complicated boundary conditions. Early in the 18th century Hodgkinson [9] recognized and pointed out the dependence of buckling load on boundary conditions for a column. However his work did little to influence the attitude of either theoretician or experimentalist in the years which followed. In both fields efforts have been concentrated around simply definable mathematical boundary conditions. Indeed, experimentalists have gone to extreme extents to duplicate these [3], despite their clear impracticability. Salmon [1] emphasized this point and stressed the need for a practical definition of the degree of boundary restraint. Today only a small amount of work is available on this question.

To obtain the exact solution to a particular eigenvalue problem, both the governing differential equations and the pertinent boundary conditions must be satisfied. However, as noted earlier, only a limited class of problems can be solved in this manner. Thus there is a need to develop techniques for obtaining approximate solutions. The energy*, finite difference, finite element** and Kantorovitch methods are foremost among the various procedures adopted.

The classical Rayleigh-Ritz [10] method (hereafter referred to as R.R.) is one of the betterknown energy techniques. The

*It can be shown that the energy method is equivalent to variational process.

**Finite element method is in fact the energy process applied in piecewise sense.

fundamental principle was proposed by Lord Rayleigh [11], and independently, from more general point of view, by Ritz. Since then, the method has been honored by their joint names and has taken a significant role in engineering, physics and mathematics. It is known that the degree of accuracy can be increased by increasing the number of independent functions employed, but this entails a great increase of labor. However when the functions are well chosen, an excellent approximation can be obtained by use of a small number of functions, [12]. The basic difficulty is to choose such approximate functions and in this matter little guidance is available.

Galerkin's method [12] and [13], belongs to the same general class as the R.R. It seeks to obtain an approximate solution of a differential equation with given boundary conditions by taking a function which satisfies these conditions exactly and proceeds to specialize the function so as to satisfy the differential equation approximately. It can be shown that if the same assumed function is employed, the R.R. and the Galerkin method give identical results.

Another widely used energy method is the method of Lagrangian multipliers [14] and [15]. This technique follows the usual R.R. method with the following two significant changes:

i) The requirement that the function satisfy the geometrical boundary conditions term by term is discarded and replaced by the requirement that the expansion as a whole satisfy the geometrical boundary conditions, and

ii) In the process of minimizing the energy of the system with respect to the coefficients, a modified energy functional that includes the Lagrangian Multipliers is used.

The first outstanding change provides much greater freedom in the selection of admissible functions. The second change converts the problem from one involving constrained variations to one involving free variations. However, if the R.R. method is easily applicable for elastically restrained structures, labor involved in the Lagrangian multiplier method is more compared to that in the former. The latter, however, enables us to obtain the lower bounds to the eigenvalues.

Modern high speed computers have made finite difference and finite element methods of impartance. These methods are capable of dealing with more general structural geometry. In practice we use regular shapes more frequently and for these, the application of the most general method cannot be justified since simpler and quicker processes will suffice.

L. V. Kantorovitch [16] proposed a method in which part of the coordinate functions are obtained from the ordinary differential equations in one variable. Very good convergence is obtained, [16] and [17], however, this process is more suitable if the resulting differential equation is readily solvable.

All methods have their region of applicability. However if a good function choice can be made, for use in the R.R. method, solutions of good accuracy are obtained with simplicity and convenient expressions for the eigenvalues can be derived.

The R.R. process is an approximation technique in which we select the approximate deflection function of the form

$$w = \sum_{i=1}^{n} a_i \varphi_i \tag{1.1}$$

where a_i are undetermined coefficients and φ_i are independent functions such that every function satisfies at least the essential boundary conditions (i.e. the prescribed geometric boundary conditions.) (see Appendix A)

The sequence is substituted into the total energy expression for the system and the function $U(a_i, \lambda)$ is obtained. This expression is then minimized with respect to the arbitrary constants and a set of linear simultaneous equations in these constants is generated. As noted earlier, the accuracy improves as the number of terms increases and the labor of computations likewise grows. The issues of concern are thus:

1. At what rate does the solution converge to the exact solution?

2. Can the choice of initial function and subsequent functions be made in such a manner that high accuracy results from the initial choice and the rate of convergence is enhanced with increasing number of terms?

This latter question is considered in detail in this thesis. It is demonstrated that for structures whose boundaries are subject to elastic restraint of varying degree, a systematic method for delineating the initial and successive functions can be devised. The process leans heavily upon synthesizing the initial displacement function from those functions which satisfy or nearly satisfy the several extreme boundary conditions i.e. these conditions in which the elastic restraint has the value either zero or infinity. It is clear from the study carried out on columns that progressive refinement can be accomplished in a systematic fashion. The errors in the solutions which are based upon the initial functions are in all cases well within the normal limits of engineering accuracy. Moreover, the form of the displacement functions chosen according to the method is such that the problems not normally tractable can be readily treated. In particular the issue of plate stability when each edge is restrained in its own individual manner becomes a relatively simple problem both for uniaxial and biaxial compression. This is demonstrated in sections which follow.

CHAPTER II

METHOD OF GENERATING DISPLACEMENT FUNCTIONS FOR ELASTICALLY RESTRAINED COLUMNS

Motivation

It can be demonstrated that, for beams under normal loading, the displacement, when the boundaries are rotationally restrained, can be expressed in the form

$$w = \frac{\sum_{i=1}^{n} f_i(\beta) f_i(\delta)}{\sum_{i=1}^{n} f_i(\beta)}$$

where the $f_i(\beta)$'s are functions solely of the restraint parameters and the $f_i(\delta)$'s are the displacement functions which correspond to extreme cases - viz clamped or simple support conditions - for the particular loading, and n is the number of the pertinent combinations of extreme restraint parameters.

For example, the displacement due to a central normal force acting on a beam of uniform EI which is elastically restrained against rotation at its ends is given by

$$w_{\beta\beta} = \frac{\beta w_{\infty\infty} + 2w_{00}}{\beta + 2} \tag{2.1}$$

where the elastic restraint parameter is defined to be $\frac{\beta EI}{\ell}$ and where $w_{\infty\infty}$ represents the displacement if the ends are clamped and w_{00} the displacement when the ends are simply supported.

It seems reasonable to suppose that a similar situation will be true, to a first approximation, in the case of columns and plates under the action of inplane forces. This supposition can be justified as follows. Consider the case of a column with equal end rotational restraints (Figure 1).

Let us assume that the approximate deflection shape for the case under consideration is given by

$$w = a_1 \varphi \tag{2.2}$$

where φ is an admissible function i.e. it satisfies at least the essential boundary conditions*. In the present case φ satisfies the conditions

$$\varphi(0) = 0 \tag{2.3a}$$

and

$$\varphi(1) = 0 \tag{2.3b}$$

Using the function (2.2), in the R.R. process, we obtain,

$$\int_0^1 (\varphi_{\xi\xi})^2 d\xi + \beta[\varphi_\xi^2(o) + \varphi_\xi^2(1)] - \lambda^2 \int_0^1 (\varphi_\xi)^2 d\xi = 0 \tag{2.4}$$

where $\beta = K\ell/EI$, and the subscripts denote differentiation. If $\varphi_\xi(0) \neq 0$ and $\varphi_\xi(1) \neq 0$, Equation (2.4) corresponds to curve (1) in Figure 2 and if $\varphi_\xi(0) = \varphi_\xi(1) = 0$, it corresponds to curve (2). The

*Prescribed geometric boundary conditions.

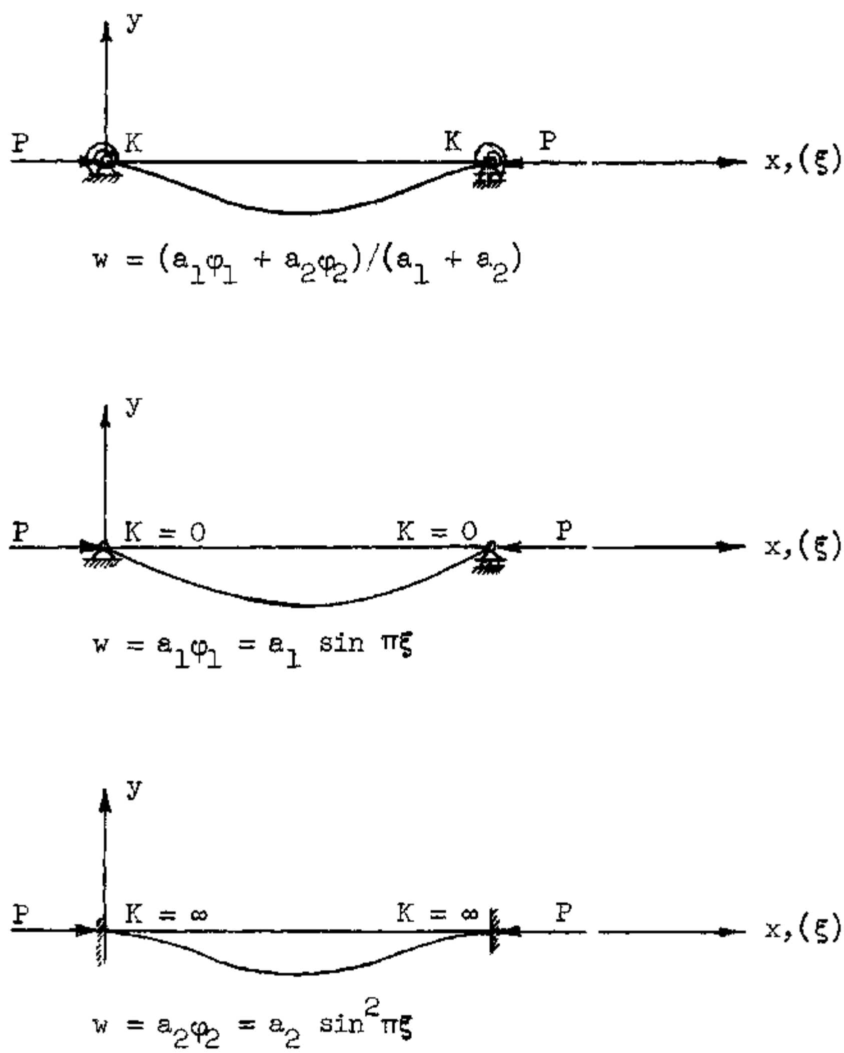


Figure 1. Column with Equal End Restraints and its Extreme Cases.

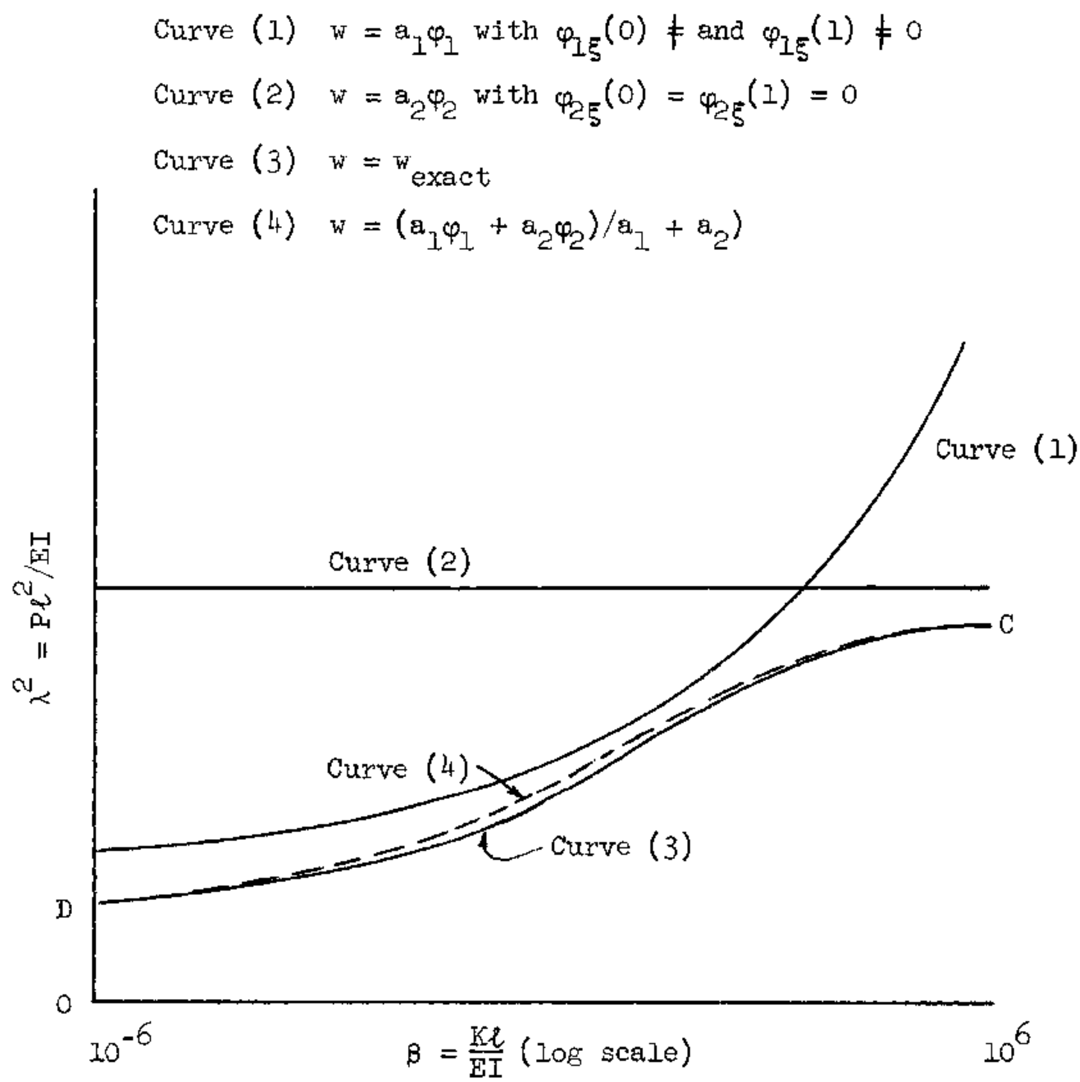


Figure 2. Buckling Coefficients VS Restraint Parameter, β, by R.R. Method for Various Function Choices.

exact solution for this problem is represented by the curve (3). It can be easily seen that either assumption, i.e. slope at the ends is zero or nonzero, gives erroneous results at the extreme values of the restraints. Hence we assume that, to a first approximation the deflection shape is given by

$$w = a_1\varphi_1 + a_2\varphi_2 \tag{2.5}$$

where

a_1 and a_2 are arbitrary constants

and

$$\varphi_{1\xi}(0) \neq 0$$

$$\varphi_{1\xi}(1) \neq 0$$

$$\varphi_{2\xi}(0) = \varphi_{2\xi}(1) = 0$$

Moreover, if φ_1 and φ_2 are chosen as exact functions corresponding to the extreme situations of the end restraints, viz. (0,0) and (∞,∞) respectively, curves (1) and (2) coincide with the true curve at these extremes (points D and C) and approximate it very closely in between. (See curve (4)).

This concept is investigated in detail first for the end restrained columns in the sections that follow.

<u>The Column with Elastic Restraints at its Ends (Figure 1)</u>

We shall deal first with the column with equal rotational restraints at its ends. In this particular situation there are two extreme cases of significance viz. both ends simply supported and both ends clamped. Thus, we begin the study by assuming that the pertinent displacement function can be taken to a first approximation as

$$w = (a_1 \sin\pi\xi + a_2 \sin^2\pi\xi)/(a_1 + a_2)^* \qquad (2.6)$$

where a_1 and a_2 are arbitrary constants.

Following the normal R.R. procedure we arrive at the condition, for a nontrivial solution

$$\begin{vmatrix} \frac{1}{2} + \frac{2\beta}{\pi^2} - \frac{\alpha}{2} & \frac{4}{3\pi}(1-\alpha) \\ \frac{4}{3\pi}(1-\alpha) & \left(2 - \frac{\alpha}{2}\right) \end{vmatrix} = 0 \qquad (2.7)$$

in which $\alpha = \dfrac{P_{cr}\,\ell^2}{\pi^2 EI}$, $\beta = \dfrac{K\ell}{EI}$ and K is the spring stiffness of the end rotational restraint. Equation (2.7) reduces to

$$\alpha = \frac{1}{2A}\{B - \sqrt{B^2 - 4AC}\} \qquad (2.8)$$

where

$$A = 2.7585$$

$$B = 4\beta + 35.1257$$

$$C = 16\beta + 32.3672$$

---

*This function has been employed by Hoff in his earlier analysis [18].

The exact characteristic equation for the column with equal restraints is

$$\beta^2\{2(1 - \cos\lambda) - \lambda \sin\lambda\} + 2\beta\lambda\{\sin\lambda - \lambda \cos\lambda\}$$
$$+ \lambda^3 \sin\lambda = 0 \tag{2.9}$$

where

$$\lambda^2 = \frac{P\ell^2}{EI} = \alpha^2\pi^2$$

The numerical solution of this equation gives results which are in full accord with those obtained from Equation (2.8). In fact the maximum discrepancy over the full range of restraints is 0.32%. (See Table 1). Such good agreement suggests that the function chosen for the displacement (Equation 2.6) must be a close approximation to the true one. This, in fact, is true as demonstrated below.

The exact displacement function for the case under consideration is given by

$$w = A\{\lambda \sin\lambda\xi + \beta(1 - \cos\lambda\xi)\} \tag{2.10}$$

and the assumed approximate function is

$$w = a_1 \sin\pi\xi + a_2 \sin^2\pi\xi \tag{2.6}$$

The constant a_1 is taken to be $2\pi/(\beta+1)$ and a_2 determined correspondingly. Using these values of the constants, the displacements are obtained from Equation (2.6) and are normalized with respect

Table 1. Buckling Coefficients for Column with Equal End Restraints, First and Second Approximation.

Critical Load $\alpha = P_{cr} \cdot \ell^2/\pi^2 EI$				
β	$w = a_1 \sin \pi\xi + a_2 \sin^2 \pi\xi$	$w = a_1 \sin \pi\xi + a_2 \sin^2 \pi\xi + a_3\left(1 - \frac{\cos k(\xi - \frac{1}{2})}{\sin k/2}\right)$	$w = a_1 \sin \pi\xi + a_2 \sin^2 \pi\xi + a_3 \sin^3 \pi\xi$	w Exact
0	1.00000	1.00000	1.00000	1.00000
0.1	1.04014	1.04014	1.04012	1.04012
1.0	1.36818	1.36816	1.36710	1.36706
5.0	2.30402	2.30397	2.29715	2.29694
10.	2.86120	2.86116	2.85417	2.85398
100	3.84522	3.84522	3.84487	3.84487
100000	3.99984	3.99984	3.99984	3.99984

to the central displacement. The comparison of these and the true displacements, Equation (2.10), are given in Table 2. It is seen that there is excellent correspondence.

Refinement in the Accuracy

The displacement function given in Equation (2.1) is a particular case of the more general expression, [19],

$$w_{\beta_1\beta_2} = \frac{\beta_1\beta_2 w_{\infty\infty} + 4\beta_1 w_{\infty 0} + 4\beta_2 w_{0\infty} + 12 w_{00}}{\beta_1\beta_2 + 4(\beta_1+\beta_2) + 12} \qquad (2.11)$$

for the case of central normal force on a beam with $\beta_1 = \beta_2 = \beta$. The simplification is a direct consequence of the fact that for the symmetrical beam with a centrally applied load $(w_{\infty 0} + w_{0\infty})$ can be expressed in terms of w_{00} and $w_{\infty\infty}$. In general this is not true and so the possibility exists that further refinement would result from a term which is proportional to the sum of the two deformations corresponding to the extreme restraints $(0,\infty)$ and $(\infty,0)$.

The deflection function is then given by

$$w = \left\{a_1 \sin\pi\xi + a_2\sin^2\pi\xi + a_3\left[1 - \frac{\cos k\left(\xi-\frac{1}{2}\right)}{\cos k/2}\right]\right\}\Big/(a_1+a_2+a_3) \qquad (2.12)$$

where

$$k = \pi\sqrt{2.0454}$$

and

$$1 - \frac{\cos k\left(\xi-\frac{1}{2}\right)}{\cos\frac{k}{2}} = w_{0\infty} + w_{\infty 0}$$

Table 2. Comparison of Assumed and True Deflection Shape for Column with Equal End Restraints.

$$w_{approx.} = \sum_{i=1}^{n} a_i \sin i\pi\xi = w_n$$

$$w_{exact} = \lambda \sin\lambda\xi + \beta(1 - \cos\lambda\xi)$$

β	ξ	w_2	w_3	w_4	w_5	w_6	w_{exact}
0	$w_{approx.} = w_{exact}$ for all values of ξ						
	0.1	0.2863	0.2882	0.2865	0.2883	0.2882	0.2883
	0.2	0.5621	0.5660	0.5638	0.5659	0.5659	0.5659
1	0.3	0.7926	0.7961	0.7943	0.7958	0.7957	0.7957
	0.4	0.9461	0.9473	0.9467	0.9472	0.9470	0.9471
	0.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.1	0.2247	0.2297	0.2274	0.2300	0.2300	0.2300
	0.2	0.4921	0.5025	0.4994	0.5023	0.5023	0.5023
5	0.3	0.7480	0.7571	0.7543	0.7563	0.7563	0.7563
	0.4	0.9327	0.9359	0.9349	0.9355	0.9355	0.9355
	0.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.1	0.1849	0.1901	0.1887	0.1905	0.1904	0.1905
	0.2	0.4470	0.4578	0.4556	0.4576	0.4576	0.4576
10	0.3	0.7192	0.7286	0.7265	0.7280	0.7279	0.7279
	0.4	0.9240	0.9273	0.9265	0.9270	0.9267	0.9269
	0.5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
∞	$w_{approx.} = w_{exact}$ for all values of ξ						

When this function is used as the basis for an energy solution, very little difference is found, as seen from Table 1. This shows that the additional nonsymmetric components have no significance in this instance.

It is noted that for the structures with elastic restraints, the Total Potential Energy includes the Energy stored in the springs. Hence we put the restriction on the nature of the additional admissible functions that there be no contribution to the energy stored in the springs. For the structures with rotational end restraints, this implies that higher approximation terms must have zero slope at the restrained boundaries. It is possible to select many such functions which are also admissible functions. However, we know that as the restraint varies from one extreme situation to the other e.g. from simply supported to clamped in the case of a column in Figure 1, the deformation shape gradually changes from one extreme to the other. The simplest form of the higher approximation function is then given by

$$w_i = a_i \varphi_1^r \varphi_2^s \tag{2.13}$$

It is easy to see that this general term has zero slope at the restrained ends if φ_1 and φ_2 are the extreme functions as defined before for a rotationally restrained structure.

Consider the case of the column with equal end restraints, Figure 1. In view of the above outlined concept, if a third term derived from the product of the deflection functions corresponding to simply supported and clamped conditions, is added a marked improvement is noted. In this case, the displacement function is

$$w = a_1 \sin \pi\xi + a_2 \sin^2\pi\xi + a_3 \sin^3\pi\xi \tag{2.14}$$

A comparison of the results for various values of β is given in Table 1.

The displacements given by Equation (2.14) and those given by the first approximation function, Equation (2.6) are compared and a substantial improvement is seen. In fact, as seen from Table 2, the displacements are very nearly exact. This strongly supports the selection of the second approximation function.

Consider now the case when further terms are added. The expanded function is of the form

$$\begin{aligned} w &= a_1 \sin \pi\xi + a_2 \sin^2\pi\xi + a_3(\sin \pi\xi)(\sin^2\pi\xi) \\ &\quad + a_4(\sin \pi\xi)^2(\sin^2\pi\xi) + a_5(\sin \pi\xi)(\sin^2\pi\xi)^2 \\ &\quad + a_6(\sin \pi\xi)^2(\sin^2\pi\xi)^2 + \cdots \\ &\quad + a_j(\sin \pi\xi)^{m_1}(\sin^2\pi\xi)^{m_2} \\ &= \sum_{r=0}^{m_1} \sum_{s=0}^{m_2} c_{rs}(\sin \pi\xi)^r (\sin^2\pi\xi)^s \\ &= \sum_{r=1}^{n} a_r \sin^r\pi\xi \end{aligned} \tag{2.15}$$

The characteristic determinant corresponding to six terms in this sequence is given in Table 3. The convergence which results from the added terms is clearly indicated in Table 4 and Figure 3. It is interesting to note that the assumed displacement shape rapidly converges to the true one as seen from Table 2 and Figure 4.

We assume, then, that in a problem for which there are only two relevant extreme conditions which correspond to independent displacement functions φ_1 and φ_2, the appropriate sequence is given by

$$w = \sum_{r=o}^{m} \sum_{s=o}^{n} C_{rs}\, \varphi_1^r \varphi_2^s \tag{2.16}$$

where C_{rs} are coefficients depending upon the restraint parameters. More specifically, for the "t" independent extreme function situation, the most general displacement function would appear to be of the form

$$w = \sum_{r=o}^{m_1} \sum_{s=o}^{m_2} \cdots\cdots \sum_{j=o}^{m_t} C_{rs} \cdots\cdots \varphi_1^r \varphi_2^s \cdots\cdots \varphi_t^j \tag{2.17}$$

However, we anticipate that, even if these complicated forms are true, the linear terms will be of the far greatest significance. This statement merely explains in words what the detailed computations summarized in Tables 2 and 4 show.

It is worthwhile at this stage to point out that the rapid convergence seems to be related to the large magnitude of the elements in the diagonal band relative to all others. This is demonstrated in Tables 5 and 6 where characteristic determinants for typical values of

Table 3. Characteristic Determinant for a Column with Equal End Restraints Corresponding to Six Terms in the Assumed Function.

$\frac{1}{2}(1-\alpha) + \frac{2\beta}{\pi^2}$	$\frac{4}{3\pi}(1-\alpha)$	$\frac{3}{8}(1-\alpha)$	$\frac{16}{15\pi}(1-\alpha)$	$\frac{5}{16}(1-\alpha)$	$\frac{32}{35\pi}(1-\alpha)$
	$\frac{1}{4}(8-2\alpha)$	$\frac{4}{5\pi}(8-2\alpha)$	$\frac{1}{4}(8-2\alpha)$	$\frac{16}{21\pi}(8-2\alpha)$	$\frac{15}{64}(8-2\alpha)$
S	Y	$\frac{9}{16}(5-\alpha)$	$\frac{32}{35\pi}(11-2\alpha)$	$\frac{15}{128}(29-5\alpha)$	$\frac{64}{35\pi}(6-\alpha)$
	M M E		$\frac{1}{8}(32-5\alpha)$	$\frac{128}{63\pi}(7-\alpha)$	$\frac{3}{32}(52-7\alpha)$
		T R I C		$\frac{25}{256}(55-7\alpha)$	$\frac{256}{231\pi}(17-2\alpha)$
					$\frac{63}{256}(28-3\alpha)$

TABLE 4 Column with Equal End Restraints. Buckling Coefficients Corresponding to the Characteristic Determinant Given in Table 3

β	P(2)	P(3)	P(4)	P(5)	P(6)	P(exact)
.00	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
.10	1.0401363969	1.0401214063	1.0401208550	1.0401208401	1.0401208401	1.0401207656
1.00	1.3681754172	1.3671007156	1.3670633882	1.3670620918	1.3670620322	1.3670615703
5.00	2.3040172458	2.2971460521	2.2969454229	2.2969390452	2.2969387770	2.2969383299
10.00	2.8611951768	2.8541734517	2.8539902866	2.8539848030	2.8539845645	2.8539842963
100.00	3.8452180326	3.8448726833	3.8448656201	3.8448654115	3.8448654115	3.8448653817
100000.00	3.9998399913	3.9998399913	3.9998399913	3.9998399913	3.9998399913	3.9998400211

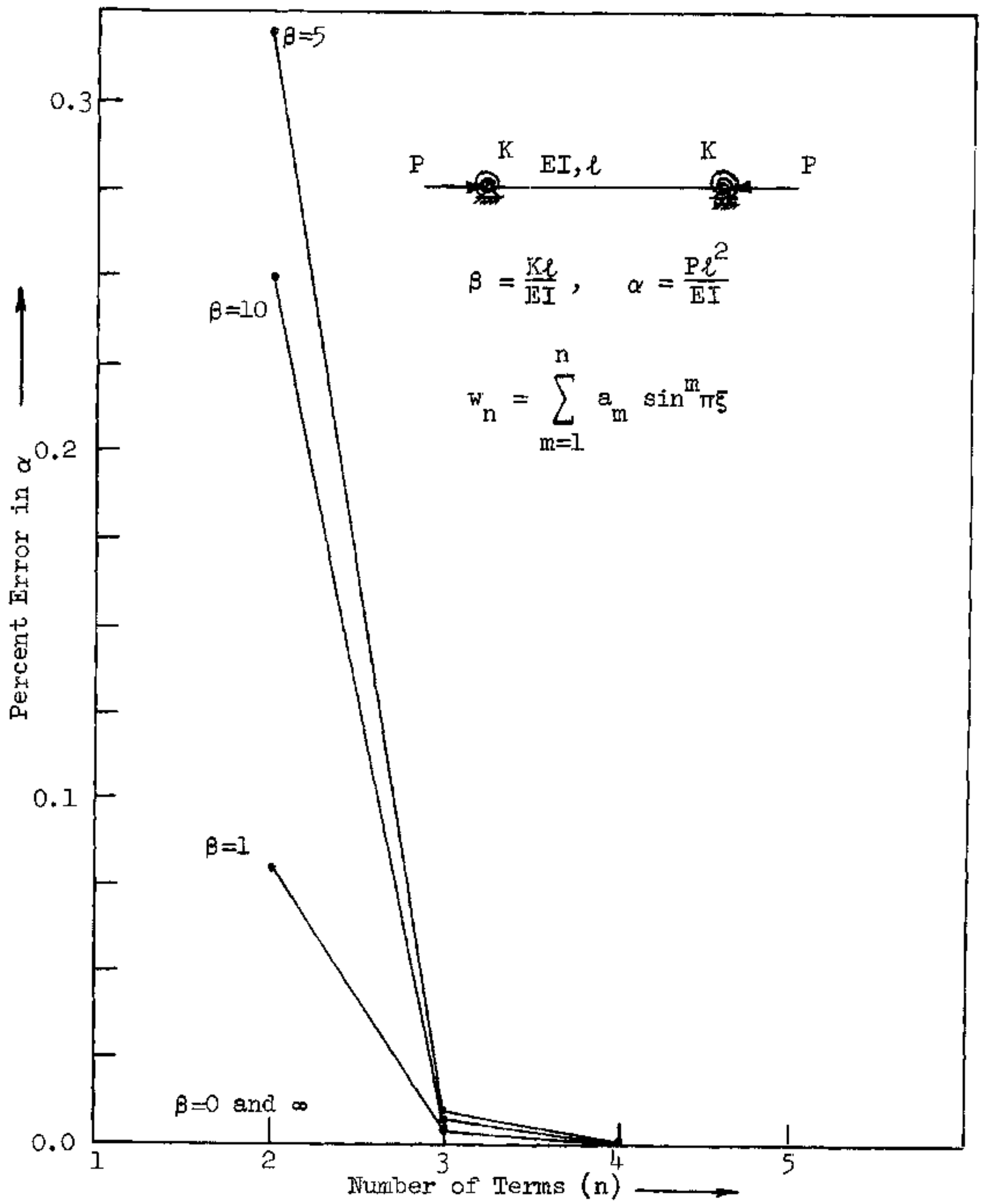


Figure 3. Percent Errors in α VS Number of Terms in the Assumed Function for a Column with Equal End Restraints.

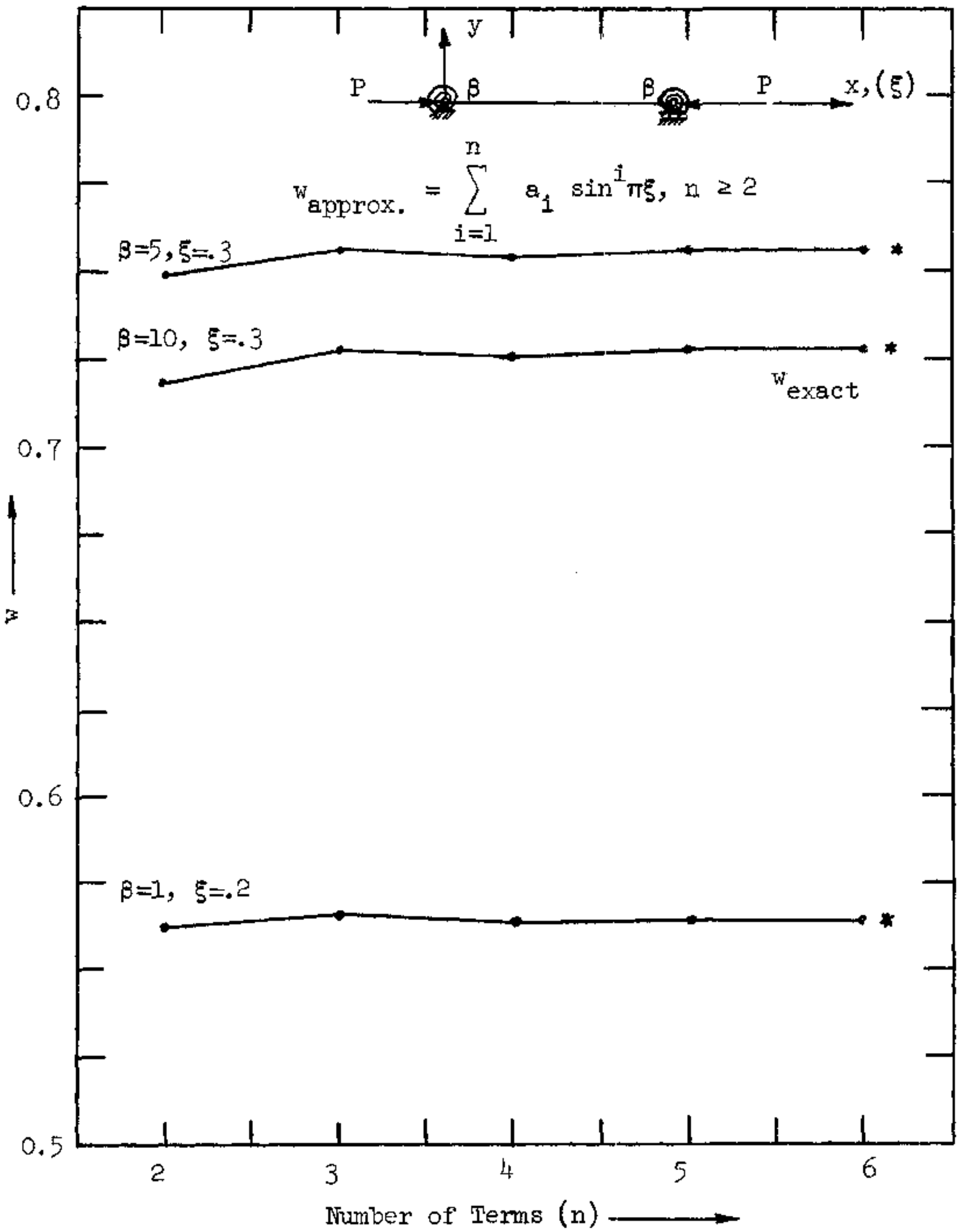


Figure 4. Convergence of the Assumed Shape to the True Shape.

Table 5. Column with Equal End Restraints. Typical Characteristic Determinant for $\beta=1$ and $\alpha_{approx.} = 1.3667$.

0.0193	-0.1556	-0.1375	-0.1245	-0.1146	-0.1067
-0.1556	1.3167	1.3411	1.3167	1.2773	1.2344
-0.1375	1.3411	2.0437	2.4058	2.5976	2.6968
-0.1245	1.3167	2.4058	3.1458	3.6432	3.9781
-0.1146	1.2773	2.5976	3.6432	4.4368	5.0300
-0.1067	1.2344	2.6968	3.9781	5.0300	5.8816

Table 5a. First Four Rows and Columns of Table 5 After Manipulations.

0.0193	-0.1556	0.0181	0.0130
-0.1556	1.3167	0.0244	-0.0244
0.0181	0.0244	0.6782	0.3865
0.0130	-0.0244	0.3865	0.3679

Table 6. Column with Equal End Restraints. Typical Characteristic Determinant for $\beta=5$ and $\alpha_{approx.} = 2.2969$.

0.3648	-0.5504	-0.4863	-0.4403	-0.4053	-0.3774
-0.5504	0.8516	0.8674	0.8516	0.8216	0.7983
-0.4863	0.8674	1.5205	1.8644	2.0526	2.1554
-0.4403	0.8516	1.8644	2.5644	3.0416	3.3677
-0.4053	0.8216	2.0526	3.0416	3.8010	4.4250
-0.3774	0.7983	2.1554	3.3677	4.4250	5.1949

Table 6a. First Four Rows and Columns of Table 6 After Manipulations.

0.3648	-0.5504	0.0641	0.0000
-0.5504	0.8516	0.0158	0.1703
0.0641	0.0158	0.6373	1.0020
0.0000	0.1703	1.0020	1.7369

β and α are presented. It is seen that the magnitude of the elements decreases as we go away from the diagonal band. In fact after few manipulations of the rows and columns, the preponderency of the elements in the diagonal band is apparent. This is demonstrated in Tables 5.a and 6.a by considering, for simplicity, 4×4 determinants from Tables 5 and 6.

The same technique gives excellent results when applied to other cases of elastically restrained columns. This is demonstrated in Table 7 in which the various relevant quantities are listed. The results given in this Table were, of course, derived using a first approximation to the displacement function. However, if additional terms, formed as in the previous case from the basic extreme functions, are used, there is marked improvement in accuracy. This is indicated clearly in Table 8 and Figure 5 for the cantilever column with end rotational restraint. However, the accuracy using only the two basic terms is well within normal engineering tolerances.

For the cantilever column with end rotational restraint, the assumed and the true displacements, normalized with respect to the tip displacement are presented in Table 9. As before, very close correspondence is indicated.

Simple Expressions for Buckling Coefficients for Restrained Columns

The success attained with this method of function selection, and in particular the high accuracy which can be obtained with the fundamental terms, leads to the thought that it might be possible to devise a displacement function which can be used directly, in a

TABLE 7 Elastically Restrained Columns. Buckling Coefficients by Rayleigh-Ritz Process (First Approximation)

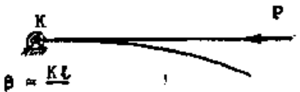
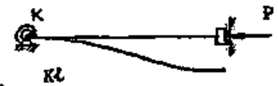
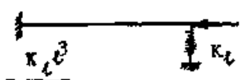
Column		 $\beta = \frac{K\ell}{EI}$	 $\beta = \frac{K\ell}{EI}$	 $\Omega = \frac{K_\ell \ell^3}{EI}$
First Approximation To Deflection Function		$a_1\xi + a_2(1-\cos\frac{\pi\xi}{2})$	$a_1\sin\frac{\pi\xi}{2} + a_2(1-\cos\pi\xi)$	$a_1(1-\cos\frac{\pi\xi}{2})$ $a_2\left\{(1-\xi) - \frac{\sin k(1-\xi)}{\sin k}\right\}$
$\alpha = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$	A	$4\pi^2 - 32$	1.3793	14.6611
	B	$\pi^2 + 4\beta$	$\beta + 4.3907$	$1.3441\ \Omega + 36.7138$
	C	β	$\beta + 1.0115$	$2.7492\ \Omega + 8.2622$
Exact Characteristic Equation		$\tan\lambda = \frac{\beta}{\lambda}$	$\tan\lambda = \frac{\lambda}{\beta}$	$\Omega(\sin\lambda - \lambda\cos\lambda) + \lambda^3\cos\lambda = 0$
Maximum Descrepancy (%)		0.2	0.31	0.47

TABLE 8 Cantilever Column with End Rotational Restraint (Figure 5) Buckling Coefficients by R. R. Process

NO. of terms / β	α = Buckling Coefficient (R.R. Method)			α Exact
	2	3	4	
0.00	0.000000	0.000000	0.000000	0.000000
0.01	0.001010	0.001010	0.001010	0.001010
0.10	0.009807	0.009805	0.009803	0.009803
0.50	0.043306	0.043267	0.043241	0.043240
1.00	0.075145	0.075057	0.074996	0.074995
10.00	0.206945	0.206898	0.206865	0.206865
100.00	0.245075	0.245075	0.245075	0.245075
100000.00	0.250000	0.250000	0.250000	0.250000

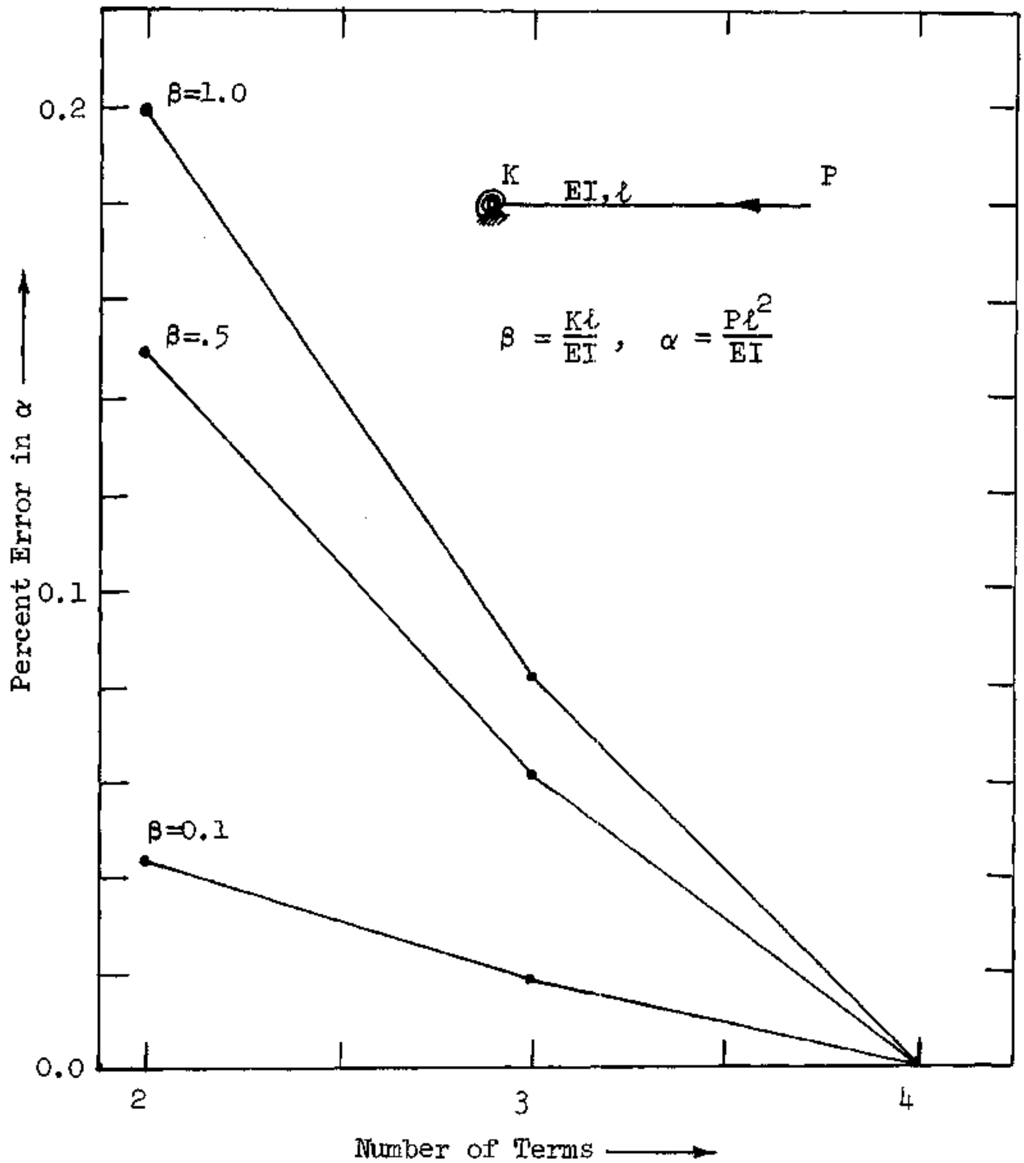


Figure 5. Percent Errors in α VS Numbers of Terms in the Assumed Function for a Cantliever Column with End Rotational Restraint.

Table 9. Comparison of Assumed and True Deflection Shape for Cantilever Column with End Rotational Restraint.

$$w_{approx.} = a_1\xi + a_2\left(1-\cos\frac{\pi\xi}{2}\right) + a_3\xi\left(1-\cos\frac{\pi\xi}{2}\right)$$

$$w_{exact} = \lambda\sin\lambda\xi + \beta(1-\cos\lambda\xi)$$

ξ	$\beta = 0.01$			$\beta = 1$		
	w_2	w_3	w_{exact}	w_2	w_3	w_{exact}
.1	0.09972	0.09971	0.09971	0.07735	0.07743	0.07762
.2	0.19952	0.19951	0.19952	0.16097	0.16143	0.16207
.3	0.29939	0.29939	0.29941	0.25065	0.25162	0.25272
.4	0.39934	0.39934	0.39936	0.34600	0.34753	0.34889
.5	0.49934	0.49936	0.49938	0.44649	0.44849	0.44988
.6	0.59940	0.59943	0.59944	0.55149	0.55378	0.55494
.7	0.69951	0.69954	0.69955	0.66022	0.66252	0.66329
.8	0.79965	0.79968	0.79968	0.77184	0.77380	0.77414
.9	0.89982	0.89984	0.89984	0.88542	0.88663	0.88665
1.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

ξ	$\beta = 5$			$\beta = 10$		
	w_2	w_3	w_{exact}	w_2	w_3	w_{exact}
.1	0.04261	0.04278	0.04304	0.03022	0.03035	0.03054
.2	0.10114	0.10118	0.10258	0.07980	0.08025	0.08084
.3	0.17500	0.17626	0.17758	0.14801	0.14890	0.14987
.4	0.26321	0.26511	0.26677	0.23367	0.23502	0.23620
.5	0.36446	0.36690	0.36859	0.33519	0.33692	0.33815
.6	0.47711	0.47987	0.48129	0.45057	0.45252	0.45357
.7	0.59922	0.60197	0.60294	0.57746	0.57940	0.58013
.8	0.72866	0.73098	0.73143	0.71325	0.71488	0.71524
.9	0.86307	0.86449	0.86455	0.85509	0.85609	0.85615
1.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

Rayleigh's Quotient*, to obtain an approximate formula for column critical loads in terms of the restraint parameter.

To do this we choose as before, for the column with equal end restraints, the deflection to be

$$w = (a_1 \sin \pi\xi + a_2 \sin^2 \pi\xi)/(a_1 + a_2) \tag{2.18}$$

where a_1 and a_2 are dependent upon the end fixity. We recognize that this function must become $\sin \pi\xi$ when the column has simply supported ends and $\sin^2 \pi\xi$ when they are clamped. Thus we deduce that a_2 is proportional to the end fixity and a_1 is independent of it. The restraint conditions are

$$w'' = -\beta\, w' \qquad \text{at } \xi = 1 \tag{2.19}$$

and

$$w'' = \beta\, w' \qquad \text{at } \xi = 0$$

and so we deduce that

$$\frac{a_1}{a_2} = \frac{2\pi}{\beta} \tag{2.20}$$

Thus

$$w = \frac{2\pi \sin \frac{\pi x}{\ell} + \beta \sin^2 \frac{\pi x}{\ell}}{2\pi + \beta} \tag{2.21}$$

*Rayleigh Quotient is the expression which results from the R.R. process when only one arbitrary constant is involved.

Using this function in a Rayleigh Quotient it is simple to demonstrate

$$P_{cr} = \frac{12\beta^2 + 80\beta + 12\pi^2}{3\beta^2 + 32\beta + 12\pi^2} \cdot \frac{\pi^2 EI}{\ell^2} = \alpha \cdot \frac{\pi^2 EI}{\ell^2} \tag{2.22}$$

This is a very simple and convenient expression which can be shown to have a maximum error of 0.8%. The other cases can be likewise treated and the results of such a study are given in Table 10.

It is interesting to note that the approximation for the critical load of the equally end restrained column, given in Equation (2.22), can be written in a slightly different from viz

$$P_{cr} = \frac{3\beta^2 \cdot P_{\infty\infty} + 16\beta(P_{\infty\infty} + P_{oo}) + 12\pi^2 \cdot P_{oo}}{3\beta^2 + 32\beta + 12\pi^2}$$

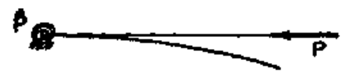
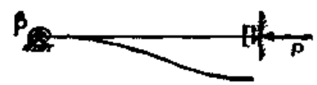
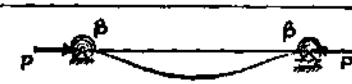
$$= \frac{(3\beta^2 + 16\beta)P_{\infty\infty} + (16\beta + 12\pi^2)P_{oo}}{(3\beta^2 + 16\beta) + (16\beta + 12\pi^2)} \tag{2.23}$$

This is of precisely the same form as the stated deflection function. It is worth observing, moreover, that some further approximation is possible. The formula can, with some slight loss in accuracy be reduced as follows:

$$P_{cr} = \frac{3(\beta + 2\pi) \cdot \beta \cdot P_{\infty\infty} + 3(\beta + 2\pi) \cdot 2\pi\, P_{oo} + 2\beta(8 - 3\pi)(P_{\infty\infty} + P_{oo})}{3(\beta + 2\pi)(\beta + 2\pi) + 4(-3\pi + 8)\beta}$$

$$= \frac{3(\beta + 2\pi)(\beta P_{\infty\infty} + 2\pi P_{oo})\left\{1 + \dfrac{2\beta \cdot (8 - 3\pi)(P_{\infty\infty} + P_{oo})}{3(\beta + 2\pi)(\beta P_{\infty\infty} + 2\pi P_{oo})}\right\}}{3(\beta + 2\pi)(\beta + 2\pi)\left\{1 + \dfrac{4\beta(8 - 3\pi)}{3(\beta + 2\pi)^2}\right\}}$$

TABLE 10 Rotationally Restrained Columns. Simple Formulae for Buckling Coefficients

		
$w = \dfrac{a_2\left\{\frac{\pi^2}{4}\xi + \beta\left(1-\cos\frac{\pi\xi}{2}\right)\right\}}{\frac{\pi^2}{4} + \beta}$ $P_{cr} = \dfrac{\pi^2 EI}{\ell^2} \cdot \dfrac{\beta^2 + 2\beta}{2(2\beta^2 + 8\beta + \pi^2)}$ or $P_{cr} = \dfrac{(2\beta^2 + 4\beta)P_\infty + (4\beta + \pi^2)P_o}{(2\beta^2 + 4\beta) + (4\beta + \pi^2)}$ which simplifies to $P_{cr} \doteq \dfrac{\beta P_\infty}{\beta + \frac{\pi}{2}}$	$w = a_2\left\{\dfrac{\pi \sin\frac{\pi\xi}{2} + \beta \sin^2\frac{\pi\xi}{2}}{\pi + \beta}\right\}$ $P_{cr} = \dfrac{12\beta^2 + 40\beta + 3\pi^2}{4(3\beta^2 + 16\beta + 3\pi^2)} \cdot \dfrac{\pi^2 EI}{\ell^2}$ or $P_{cr} = \dfrac{(3\beta^2 + 8\beta)P_\infty + (8\beta + 3\pi^2)P_o}{(3\beta^2 + 8\beta) + (8\beta + 3\pi^2)}$ which simplifies to $P_{cr} \doteq \dfrac{\beta P_\infty + \pi P_o}{\beta + \pi}$	$w = a_2\left\{\dfrac{2\pi\sin\pi\xi + \beta\sin^2\pi\xi}{\beta + 2\pi}\right\}$ $P_{cr} = \dfrac{12\beta^2 + 80\beta + 12\pi^2}{3\beta^2 + 32\beta + 12\pi^2} \cdot \dfrac{\pi^2 EI}{\ell^2}$ or $P_{cr} = \dfrac{(3\beta^2 + 16\beta)P_{\infty\infty} + (16\beta + 12\pi^2)P_{oo}}{(3\beta^2 + 16\beta) + (16\beta + 12\pi^2)}$ which simplifies to $P_{cr} \doteq \dfrac{\beta P_{\infty\infty} + 2\pi P_{oo}}{(\beta + 2\pi)}$

It is clear that for all values of β the expressions

$$\frac{2\beta(8 - 3\pi)(P_{\infty\infty} + P_{oo})}{3(\beta + 2\pi)(\beta P_{\infty\infty} + 2\pi P_{oo})} \quad \text{and} \quad \frac{4(8 - 3\pi)\beta}{3(\beta + 2\pi)^2}$$

are negligible compared to unity, and so we may write

$$P_{cr} = \frac{\beta P_{\infty\infty} + 2\pi P_{oo}}{\beta + 2\pi} \tag{2.24}$$

This is a very simple expression and has a similar form to Equation (2.21). The results of the above technique when applied to other cases are summarized in Table 10.

<u>The Column with Unequal Restraints at the Ends (Figure 6)</u>

In accordance with our previous formulation, we choose the deflection function to be

$$w = \sum_{r=o}^{m_1} \sum_{s=o}^{m_2} \sum_{j=o}^{m_3} \sum_{t=o}^{m_4} C_{rsjt} \, \varphi_1^r \varphi_2^s \varphi_3^j \varphi_4^t \tag{2.25}$$

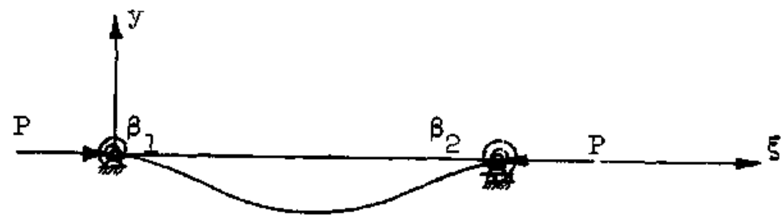
where

$$\varphi_1 = w_{oo}$$

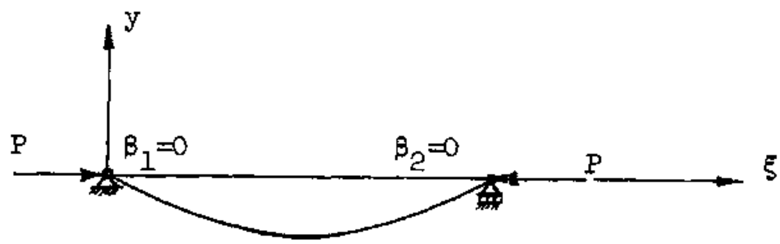
$$\varphi_2 = w_{\infty\infty}$$

$$\varphi_3 = w_{o\infty}$$

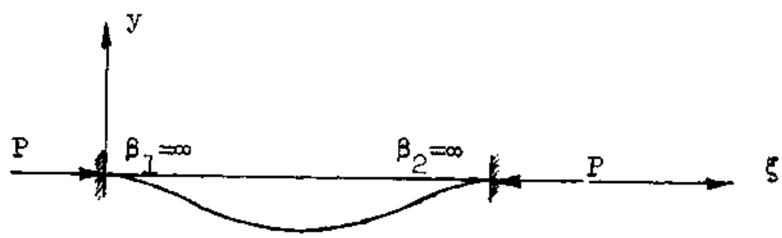
$$\varphi_4 = w_{\infty o}$$



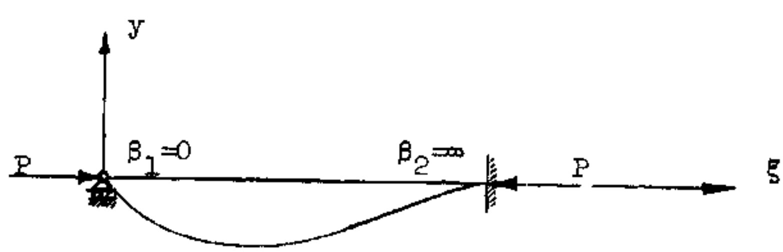
$$w = (a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3)/(a_2 + a_2 + a_3)$$



$$w = a_1\varphi_1 = a_1 \sin \pi\xi$$



$$w = a_2\varphi_2 = a_2 \sin^2 \pi\xi$$



$$w = a_3\varphi_3 = a_3\left(\xi - \frac{\sin k\xi}{\sin k}\right)$$

Figure 6. Column with Unequal End Restraints and its Extreme Cases.

Considering only the linear terms, we get,

$$w = a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3 + a_4\varphi_4$$

However, we know that in the corresponding flexure case the displacement functions w_{OO}, $w_{O\infty}$, w_{OO} and $w_{\infty O}$ are interrelated. For example, for a central load,

$$3w_{\infty\infty} + w_{OO} = 2(w_{O\infty} + w_{\infty O})$$

Thus the general displacement expression can be rewritten as

$$w = a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3 \tag{2.26}$$

For the column with unequal restraints the above function is given by

$$w = a_1 \sin \pi\xi + a_2 \sin^2\pi\xi + a_3\left(\xi - \frac{\sin k\xi}{\sin k}\right) \tag{2.27}$$

Using this function in the energy method as before, we arrive at the condition,

$$\begin{vmatrix} \frac{(1-\alpha)}{2} + \frac{\beta_1+\beta_2}{\pi^2} & \frac{4}{3\pi}(1-\alpha) & b_3\left(\frac{\alpha-1}{2}\right) + \frac{d_1\beta_1}{\pi^3} \\ \frac{4}{3\pi}(1-\alpha) & 2 - \frac{\alpha}{2} & \frac{1}{4}(\alpha b_5 - b_2) \\ b_3\left(\frac{\alpha-1}{2}\right) + \frac{d_1\beta_1}{\pi^3} & \frac{1}{4}(\alpha b_5 - b_2) & (b_1 - \alpha b_4) \end{vmatrix} = 0 \tag{2.28}$$

where

$$k = k_1 \pi \qquad = \quad \pi \sqrt{2.0454}$$

$$b_1 = \frac{k_1^{\,4}}{2} \qquad = \quad 2.0920$$

$$b_2 = \frac{8\pi\, k_1^{\,3}(1-\cos k)}{\sin k(4\pi^2 - k^2)} \qquad = - \, 4.7544$$

$$b_3 = \frac{2\pi\, k_1^{\,2}}{(\pi^2 - k^2)} \qquad = - \, 1.2455$$

$$b_4 = \frac{k_1^{\,2}}{2} \qquad = \quad 1.0230$$

$$b_5 = \frac{b_2}{k_1^{\,2}} \qquad = - \, 2.3244$$

$$d_1 = 1 - \frac{k}{\sin k} \qquad = \quad 5.6033$$

Equation (2.28) was solved numerically. The results are displayed in Table 11 and maximum discrepancy is seen to be 0.33 percent.

The earlier analysis of Hoff [18] gives strong support to the method of function selection outlined. Hoff, for his analysis, chose the deflection shape to be

$$w = A \sin \pi\xi + \frac{B}{2}\,(1 - \cos 2\pi\xi) + C \sin 2\pi\xi \tag{2.29}$$

It is readily seen that this differs from the one adopted here only in the third term. This difference, however, has substantial influence

Table 11. Column with Unequal End Restraints. Comparison of Buckling Coefficients by Rayleigh-Ritz Method (1st Approximation) and by Solving Exact Characteristic Equation.

0.0	1.0	5.0	10.0	100.0	10^5	β_1 / β_2
1.0000 1.0000 (0.0000)	1.1748 1.1755 (0.0603)	1.5475 1.5492 (0.1082)	1.7305 1.7310 (0.0288)	2.0056 2.0056 (0.0000)	2.0454 2.0454 (0.0000)	0.0
	1.3667 1.3682 (0.1076)	1.7799 1.7812 (0.0701)	1.9819 1.9828 (0.0465)	2.2841 2.2850 (0.0398)	2.3275 2.3284 (0.0405)	1.0
		2.2973 2.3040 (0.2913)	2.5577 2.5628 (0.2009)	2.9431 2.9512 (0.2754)	2.9967 3.0065 (0.3271)	5.0
Exact R.R. (% Error)			2.8542 2.8612 (0.2445)	3.2977 3.3048 (0.2164)	3.3589 3.3680 (0.2718)	10.0
				3.8448 3.8452 (0.0109)	3.9207 3.9215 (0.0203)	100.0
					3.9999 3.9999 (0.0000)	10^5

when one restraint is small and other large. This is due to the fact that the third term has nonzero slope at $\xi = 0$ and 1 and the assumed shape poorly approximates the true shape when $\beta_1 = 0$ and $\beta_2 = \infty$. The error in Hoff's solution is 7.9 percent whereas the maximum error in the solutions given in this thesis is .32 percent. This is demonstrated in Table 12 and Figure 7.

It has been shown earlier that using the independent extreme functions it is possible to develop a function which can be employed in the Rayleigh's Quotient to obtain an expression for the critical load. Thus with the deflection function given by Equation (2.27) and satisfying the natural boundary conditions at the restrained ends, viz.

$$w'' = -\beta_2 w' \quad \text{at} \quad \xi = 1$$

and

$$w'' = \beta_1 w' \quad \text{at} \quad \xi = 0$$

we obtain,

$$w = \Big\{f_1(\beta_1\beta_2)\sin\pi\xi + f_2(\beta_1\beta_2)\sin^2\pi\xi$$

$$+ f_3(\beta_1\beta_2)\left(\xi - \frac{\sin k\xi}{\sin k}\right)\Big\}\Big/(f_1+f_2+f_3) \qquad (2.30)$$

where

$$f_1(\beta_1\beta_2) = 2\pi$$

$$f_2(\beta_1\beta_2) = \beta_1(\beta_2 d_1 + k^2)/(\beta_1 d_1 + k^2)$$

Table 12. Column with Unequal End Restraints. Comparison of Buckling Coefficients Obtained by Hoff [18], and Exact Coefficients.

0	1	5	10	100	10^5	β_1 / β_2
1.0000 1.0000 (0.0000)	1.1748 1.1807 (0.5029)	1.5475 1.6029 (3.5810)	1.7305 1.8242 (5.4170)	2.0056 2.1600 (7.6972)	2.0454 2.2066 (7.8807)	0
	1.3667 1.3682 (0.1079)	1.7799 1.8083 (1.5940)	1.9819 2.0388 (2.8718)	2.2841 2.3850 (4.4170)	2.3275 2.4333 (4.5446)	1
		2.2973 2.3040 (0.2923)	2.5577 2.5694 (0.4581)	2.9431 2.9671 (0.8150)	2.9967 3.0217 (0.8327)	5
Exact			2.8542 2.8612 (0.2448)	3.2977 3.3055 (0.2361)	3.3589 3.3663 (0.2189)	10
Hoff [18] % Error				3.8448 3.8452 (0.0107)	3.9207 3.9207 (0.0000)	100
					3.9999 3.9999 (0.0000)	10^5

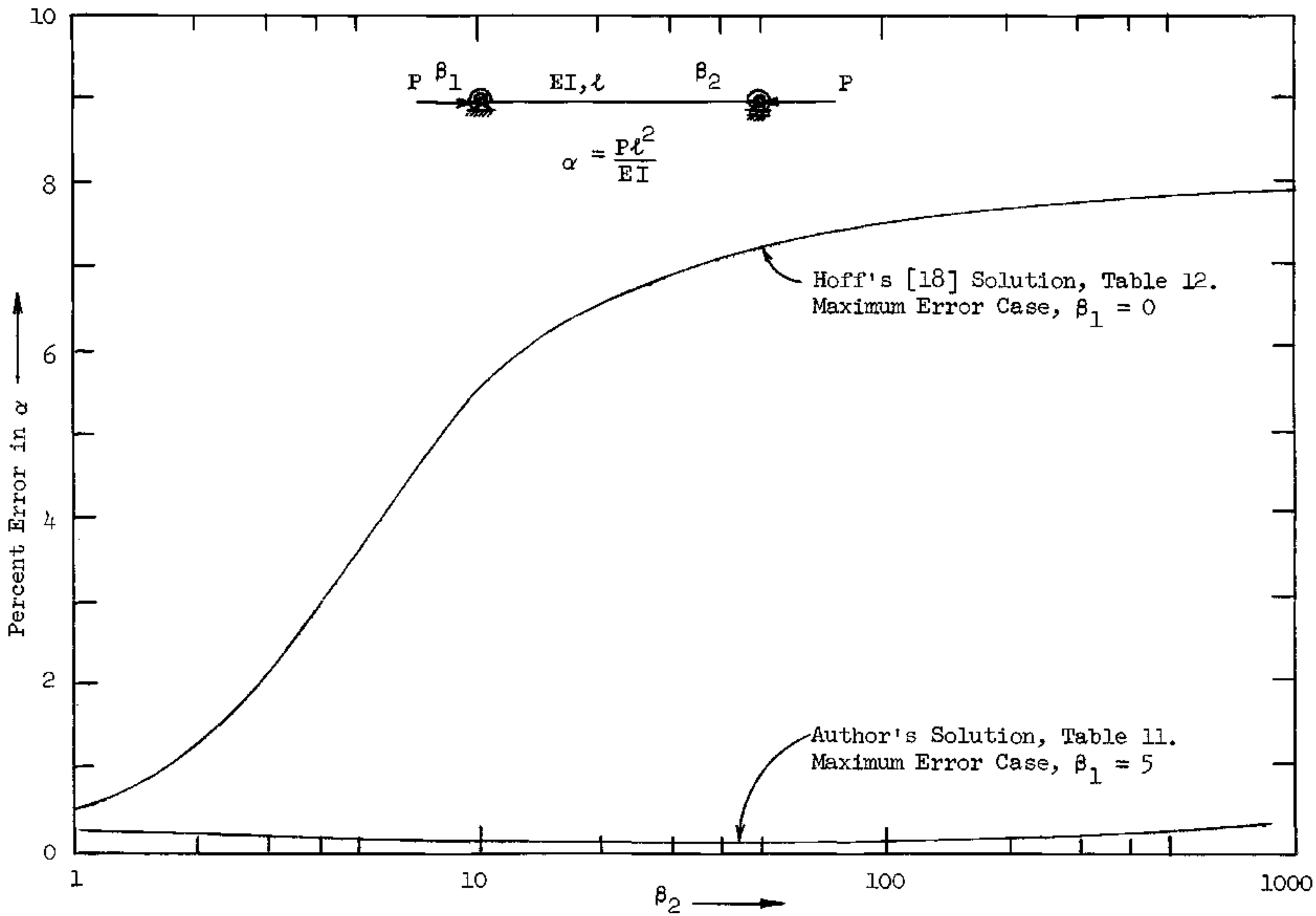


Figure 7. Graphical Presentation of Maximum Percent Error Case in Tables (11) and (12).

$$f_3(\beta_1\beta_2) = 2\pi^2(\beta_2-\beta_1)/(\beta_1 d_1 + k^2)$$

Thus, the critical buckling efficient, α, is given by

$$\alpha = \frac{g_1(\beta_1,\beta_2)}{g_2(\beta_1,\beta_2)} \tag{2.31}$$

where

$$g_1(\beta_1,\beta_2) = 62.7948\ \beta_1^2\beta_2^2 + 482.5776\ \beta_1^2\beta_2 + 388.519\ \beta_1\beta_2^2$$
$$+ 2595.92\ \beta_1\beta_2 + 1040.2583\ \beta_1^2 + 815.051\ \beta_2^2$$
$$+ 5150.58\ \beta_1 + 4748.72\ \beta_2 + 8044.21$$

$$g_2(\beta_1,\beta_2) = 15.6987\ \beta_1^2\beta_2^2 + 152.022\ \beta_1^2\beta_2 + 128.546\ \beta_1\beta_2^2$$
$$+ 1135.07\ \beta_1\beta_2 + 496.541\ \beta_1^2 + 398.48\ \beta_2^2$$
$$+ 3520.48\ \beta_1 + 3118.62\ \beta_2 + 8044.21$$

This expression can be simplified as

$$\alpha \cong \frac{(4\beta_1\beta_2 + 25\beta_2 + 4\pi^2)(3\beta_1\beta_2 + 25\beta_1 + 6\beta_2 + 38)\left\{1 + \frac{A(\beta_1,\beta_2)}{g_1^*(\beta_1,\beta_2)}\right\}}{(\ \beta_1\beta_2 + 12\beta_2 + 4\pi^2)(3\beta_1\beta_2 + 25\beta_1 + 6\beta_2 + 38)\left\{1 + \frac{B(\beta_1,\beta_2)}{g_2^*(\beta_1,\beta_2)}\right\}} \tag{2.32}$$

where

$$\frac{A(\beta_1,\beta_2)}{g_1^*(\beta_1,\beta_2)}$$

$$\approx \frac{-\ 8\beta_1^2\beta_2 - 25\beta_1\beta_2^2 - 400\beta_1\beta_2 + 199\beta_1^2 + 6\beta_2^2 - 3\beta_1 - 279\beta_2 + 37}{12\beta_1^2\beta_2^2 + 100\beta_1^2\beta_2 + 99\beta_1\beta_2^2 + 896\beta_1\beta_2 + 150\beta_2^2 + 987\beta_1 + 1187\beta_2 + 1500}$$

and

$$\frac{B(\beta_1,\beta_2)}{g_2^*(\beta_1,\beta_2)}$$

$$\approx \frac{4\beta_1^2\beta_2 - 17\beta_1\beta_2^2 - 239\beta_1\beta_2 + 95\beta_1^2 + 4\beta_2^2 - 313\beta_1 - 97\beta_2 + 37}{3\beta_1^2\beta_2^2 + 25\beta_1^2\beta_2 + 42\beta_1\beta_2^2 + 456\beta_1\beta_2 + 72\beta_2^2 + 986\beta_1 + 693\beta_2 + 1500}$$

It can be seen that, in the range of the restraints under consideration (viz. $\beta_1 \leq \beta_2$),

$$\frac{A}{g^*(\beta_1,\beta_2)} \ll 1 \quad \text{and} \quad \frac{B}{g^*(\beta_1,\beta_2)} \ll 1$$

Hence the Equation (2.31) reduces to

$$\alpha_{\beta_1,\beta_2} \simeq \frac{\alpha_{\infty\infty}\beta_1\beta_2 + 12\beta_2\alpha_{0\infty} + 4\pi^2\alpha_{00}}{\beta_1\beta_2 + 12\beta_2 + 4\pi^2} \tag{2.33}$$

which is a very convenient expression. This expression has the maximum error of 9.5 percent.

The principles discussed in this chapter have been combined with allied concepts to deal with more general situations such as plate buckling. This work is reported in the next chapter.

CHAPTER III

EXTENSION TO PLATES

The method outlined in the previous chapter can be extended readily to deal with an elastically restrained plate. We shall consider first a uniaxially compressed plate with simply supported loaded edges but with equal uniform elastic restraints along its unloaded sides. The initial deflection function will be taken as

$$w = \frac{a_1 f_1 + a_2 f_2}{a_1 + a_2} \tag{3.1}$$

where f_1 and f_2 are defined in Figures (8a) and (8b).

When this function is used with a normal Rayleigh-Ritz procedure we arrive at the following expressions connecting the two arbitrarily chosen constants a_1 and a_2,

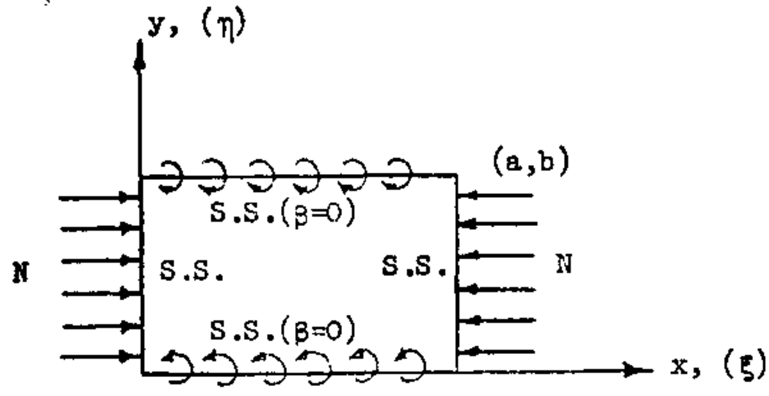
$$a_1\left\{\frac{(m^2+\gamma^2)^2}{4} + \frac{\beta\gamma^4}{\pi^2} - \mu\,\frac{m^2\gamma^2}{4}\right\} + a_2\{(m^2+\gamma^2)^2 - \mu m^2\gamma^2\}\frac{2}{3\pi} = 0 \tag{3.2a}$$

and

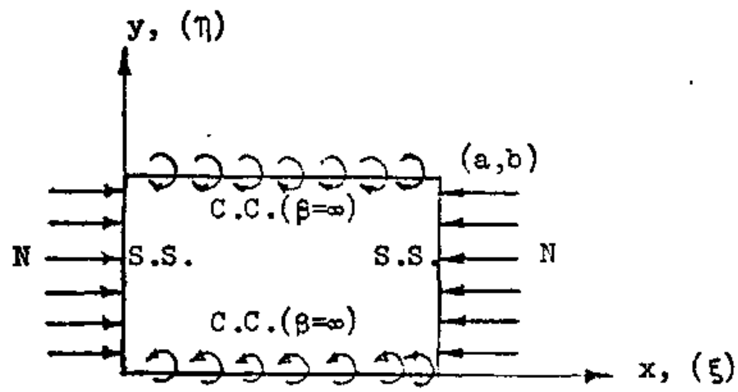
$$a_1\{(m^2+\gamma^2)^2 - \mu m^2\gamma^2\}\frac{2}{3\pi}$$

$$+ a_2 \cdot \frac{1}{16}\{3m^4 + 8m^2\gamma^2 + 16\gamma^4 - 3\mu m^2\gamma^2\} = 0 \tag{3.2b}$$

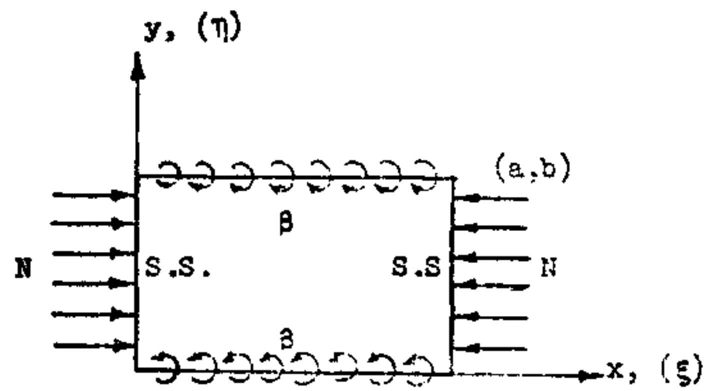
where γ is the plate aspect ratio.



$$w = a_1 f_1 = a_1 \sin m\pi\xi \sin \pi\eta$$



$$w = a_2 f_2 = a_2 \sin m\pi\xi \sin^2 \pi\eta$$



$$w = \frac{a_1 f_1 + a_2 f_2}{a_1 + a_2}$$

Figure 8. Rectangular Plate with Simply Supported Loaded Edges, Rotationally Restrained Unloaded Edges and its Extreme Cases.

m is the number of halfwaves in longitudinal direction

and

$$\mu = \frac{Nb^2}{\pi^2 D}$$

For a nontrivial solution the determinant of the coefficients must vanish. As a consequence a quadratic equation in μ is generated. When the buckling coefficients determined in this manner are compared with those derived in reference [20]*, excellent agreement is obtained. This is clear from Table 13.

We turn now to consider the convergence which results adding further terms to the previous displacement function. Consistent with the process previously adopted for the column we choose the deflection to be of the form

$$w = \sin m\pi\xi \sum_{r=1}^{n} a_r \sin^r \pi\eta \tag{3.3}$$

After the normal algebraic tedium we arrive at the expression,

$$\sum_{r=1}^{n} a_r \Big\{\frac{m^4}{2} C_{1,M} - m^2\gamma^2 r[(r-1)C_{2,M} - C_{1,M}]$$

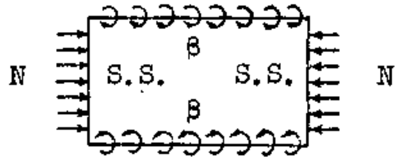
$$+ \gamma^4 \frac{r}{2} [(r-1)(r-2)(r-3)C_{3,M} - 2(r-1)(3r-4)C_{2,M}$$

$$- (2-3r)C_{1,M}] - \gamma^2\mu \frac{m^2}{2} C_{1,M}\Big\} - a_2\gamma^4 \frac{2}{\pi}$$

---

*These are Lundquist's and Stowel's recommended results obtained by applying corrections to the results of Energy Method.

Table 13. Buckling Coefficients, μ, Obtained by Solving Equation (3.4).



β	$\left(\frac{m}{\gamma}\right)$	μ_2	μ_3	μ_4	μ_5	μ_6	$\mu_{Ref.(20)}$
0	1/.6	5.1377	5.1377	5.1377	5.1377	5.1377	5.1377
0	1/.8	4.2025	4.2025	4.2025	4.2025	4.2025	4.2025
0	1/1	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000
0	1/1.2	4.1344	4.1344	4.1344	4.1344	4.1344	4.1344
0	1/1.4	4.4702	4.4702	4.4702	4.4702	4.4702	4.4702
0	1/1.6	4.9506	4.9506	4.9506	4.9506	4.9506	4.9506
1	1/.6	5.2750	5.2737	5.2735	5.2735	5.2735	5.2710
1	1/.8	4.4443	4.4424	4.4422	4.4422	4.4422	4.4410
1	1/1	4.3759	4.3733	4.3732	4.3731	4.3731	4.3730
1	1/1.2	4.6741	4.6707	4.6705	4.6705	4.6705	4.6700
1	1/1.4	5.2033	5.1990	5.1987	5.1987	5.1987	5.1990
1	1/1.6	5.9069	5.9015	5.9012	5.9012	5.9012	5.9010
5	1/.6	5.6913	5.6701	5.6683	5.6682	5.6682	5.6660
5	1/.8	5.1518	5.1231	5.1212	5.1211	5.1211	5.1200
5	1/1	5.4538	5.4164	5.4143	5.4142	5.4142	5.4142
5	1/1.2	6.2029	6.1552	6.1526	6.1525	6.1525	6.1510
5	1/1.6	8.5809	8.5072	8.5035	8.5034	8.5033	8.5020
5	1/1.8	10.1265	10.0371	10.0327	10.0325	10.0325	--

μ_i = Buckling Coefficient Corresponding to i Terms in the Assumed Function.

Table 13. Buckling Coefficients, μ, Obtained by Solving Equation (3.4). (Continued)

β	$\left(\frac{m}{\gamma}\right)$	μ_2	μ_3	μ_4	μ_5	μ_6	$\mu_{Ref.(20)}$
10	1/.6	6.0277	5.9747	5.9705	5.9702	5.9702	5.9740
10	1/.8	5.6958	5.6273	5.6229	5.6227	5.6227	5.6250
10	1/1	6.2606	6.1735	6.1685	6.1683	6.1682	6.1690
10	1/1.2	7.3291	7.2196	7.2137	7.2134	7.2134	7.2120
10	1/1.4	8.7669	8.6309	8.6238	8.6235	8.6235	8.6230
10	1/1.6	10.5179	10.3512	10.3429	10.3425	10.3425	--
100	1/.6	7.0784	6.8541	6.8369	6.8359	6.8358	6.8450
100	1/.8	7.2521	6.9956	6.9792	6.9783	6.9783	6.9810
100	1/1	8.4681	8.1623	8.1446	8.1437	8.1437	8.1370
100	1/1.2	10.3334	9.9630	9.9428	9.9419	9.9418	--
100	1/1.4	12.7133	12.2643	12.2408	12.2397	12.2396	--
100	1/1.6	15.5516	15.0106	14.9829	14.9817	14.9816	--
10^5	1/.6	7.3641	7.0780	7.0563	7.0550	7.0550	7.0590*
10^5	1/.8	7.6420	7.3247	7.3044	7.3034	7.3033	7.3040
10^5	1/1	8.9994	8.6267	8.6050	8.6040	8.6039	8.5930
10^5	1/1.2	11.0403	10.5928	10.5683	10.5671	10.5671	--
10^5	1/1.4	13.6292	13.0896	13.0611	13.0598	13.0597	--
10^5	1/1.6	16.7093	16.0614	16.0280	16.0265	16.0264	--

*These values in Ref. (20) are corresponding to $\beta = \infty$.

$$+ a_1\gamma^4 \frac{\beta}{\pi^2} = 0$$

$$\text{for } s = 1, 2, 3, \ldots. \qquad (3.4)$$

where

$$M = r + s$$

and the last two terms are present only for s = 1.

The coefficients $C_{i,M}$ are defined as follows

$$C_{1,M} = \int_o^1 \sin^M \pi\xi d\xi = \frac{M-1}{M} \times \frac{M-3}{M-2} \times \cdots\cdots\cdots \times \frac{3}{4} \times \frac{1}{2}, \text{ M even}$$

$$= \frac{M-1}{M} \times \frac{M-3}{M-2} \times \cdots\cdots\cdots\cdots\cdots \times \frac{4}{5} \times \frac{2}{3} \times \frac{2}{\pi}, \text{ M odd}$$

$$C_{2,M} = \int_o^1 \sin^{M-2}\xi \cdot \cos^2\pi\xi = \frac{1}{M} \times \frac{M-3}{M-2} \times \frac{M-5}{M-4} \times \cdots\cdots\cdots \frac{3}{4} \times \frac{1}{2}, \text{ M even}$$

$$= \frac{1}{M} \times \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \frac{2}{3} \times \frac{2}{\pi}, \text{ M odd}$$

$$C_{3,M} = \int_o^1 \sin^{M-4}\pi\xi \cdot \cos^4\pi\xi d\xi = \frac{3}{m} \times \frac{1}{M-2} \times \frac{M-5}{M-4} \times \cdots\cdots\cdots \frac{3}{4} \times \frac{1}{2}, \text{ M even}$$

$$= \frac{3}{M} \times \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \frac{2}{3} \times \frac{2}{\pi}, \text{ M odd}$$

The results clearly indicate that if only the two characteristic terms are used the values are well within engineering accuracy. However, when the subsequent generated terms are taken into account,

$$+ 2m^2\gamma^2 \cdot \sum_r a_r\, r^2[(r-1)D_{2,M} - D_{1,M}][(r-1)C_{2,M} - C_{1,M}]$$

$$+ \gamma^4 \sum_r a_r\, D_{1,M}\{(r-1)(r-2)(r-3) \cdot C_{3,M} - 2(r-1)(3r-4)C_{2,M}$$

$$- (2-3r)C_{1,M}\} + \mu\gamma^2 m^2 \sum_r a_r \frac{r}{2} [(r-1)D_{2,M} - D_{1,M}]$$

$$+ \mu\gamma^4 \sum_r a_r\, r\, D_{1,M}[C_{2,M}(r-1) - C_{1,M}] + \frac{a_1\beta}{\pi^2} (m^2\gamma + \gamma^4)$$

$$- a_2 K_m \frac{8}{3\pi^2} \left(m^3 + \frac{\gamma^4}{m}\right) = 0$$

$$\text{for } s = 1,2,3,\cdots \qquad (3.6)$$

where

$$M = r + s$$

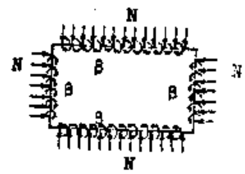
$$K_m = [1 - (-1)^m]$$

and last two terms are present only for $s = 1$. The term $D_{i,M}$ is defined to be $\frac{C_{i,M}}{q}$ where q has the value of unity when M is even and m when M is odd.

The solutions are given in Table 14 and are displayed in Figure 9. As before, the convergence is outstanding. In a latter part of this chapter, Rayleigh's Quotient is derived for this case, using a function formed as a product of true vibration mode shapes of

Table 14. Buckling Coefficients, μ, Obtained by Solving Equation (3.6).

μ_i = Buckling coefficient, μ, corresponding to i terms.



β	γ	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7
.00	.60	3.777778	3.777778	3.777778	3.777778	3.777778	3.777778
.00	.80	2.562500	2.562500	2.562500	2.562500	2.562500	2.562500
.00	1.00	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000
.00	1.20	1.694444	1.694444	1.694444	1.694444	1.694444	1.694444
.00	1.40	1.510204	1.510204	1.510204	1.510204	1.510204	1.510204
.00	1.60	1.390625	1.390625	1.390625	1.390625	1.390625	1.390625
.00	1.80	1.308642	1.308642	1.308642	1.308642	1.308642	1.308642
.00	2.00	1.250000	1.250000	1.250000	1.250000	1.250000	1.250000
.00	2.20	1.206612	1.206612	1.206612	1.206612	1.206612	1.206612
.00	2.40	1.173611	1.173611	1.173611	1.173611	1.173611	1.173611
.00	2.60	1.147929	1.147929	1.147929	1.147929	1.147929	1.147929
.00	2.80	1.127551	1.127551	1.127551	1.127551	1.127551	1.127551
.00	3.00	1.111111	1.111111	1.111111	1.111111	1.111111	1.111111
.00	3.20	1.097656	1.097656	1.097656	1.097656	1.097656	1.097656
.00	3.40	1.086505	1.086505	1.086505	1.086505	1.086505	1.086505
.00	3.60	1.077160	1.077160	1.077160	1.077160	1.077160	1.077160
.00	3.80	1.069252	1.069252	1.069252	1.069252	1.069252	1.069252

β	γ	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7
.00	4.00	1.062500	1.062500	1.062500	1.062500	1.062500	1.062500
.10	.60	3.837834	3.837809	3.837808	3.837808	3.837808	3.837808
.10	.80	2.608868	2.608840	2.608839	2.608839	2.608839	2.608839
.10	1.00	2.040189	2.040161	2.040159	2.040159	2.040159	2.040159
.10	1.20	1.731866	1.731838	1.731837	1.731837	1.731836	1.731836
.10	1.40	1.546482	1.546456	1.546455	1.546455	1.546455	1.546455
.10	1.60	1.426546	1.426521	1.426520	1.426520	1.426520	1.426520
.10	1.80	1.344583	1.344560	1.344558	1.344558	1.344558	1.344558
.10	2.00	1.286134	1.286112	1.286111	1.286111	1.286111	1.286111
.10	2.20	1.243011	1.242990	1.242989	1.242989	1.242989	1.242989
.10	2.40	1.210297	1.210276	1.210275	1.210275	1.210275	1.210275
.10	2.60	1.184896	1.184875	1.184874	1.184874	1.184874	1.184874
.10	2.80	1.164782	1.164762	1.164761	1.164761	1.164761	1.174761
.10	3.00	1.148586	1.148566	1.148565	1.148565	1.148565	1.148565
.10	3.20	1.135352	1.135333	1.135332	1.135332	1.135332	1.135331
.10	3.40	1.124401	1.124381	1.124380	1.124380	1.124380	1.124380
.10	3.60	1.115235	1.115216	1.115215	1.115215	1.115215	1.115215
.10	3.80	1.107488	1.107469	1.107468	1.107468	1.107468	1.107468
.10	4.00	1.100881	1.100862	1.100861	1.100861	1.100861	1.100861

β	γ	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7
1.00	.60	4.348746	4.346699	4.346587	4.346583	4.346583	4.346582
1.00	.80	2.996919	2.994700	2.994573	2.994568	2.994568	2.994568
1.00	1.00	2.372793	2.370555	2.370424	2.370420	2.374020	2.370419
1.00	1.20	2.039716	2.037586	2.037464	2.037460	2.037460	2.037459
1.00	1.40	1.844140	1.842157	1.842047	1.842043	1.843043	1.842043
1.00	1.60	1.720983	1.719135	1.719036	1.719033	1.719033	1.719032
1.00	1.80	1.639108	1.637367	1.637277	1.637274	1.637274	1.637274
1.00	2.00	1.582256	1.580598	1.580515	1.580511	1.580511	1.580511
1.00	2.20	1.542345	1.539750	1.539671	1.539668	1.539667	1.539667
1.00	2.40	1.511015	1.509468	1.509392	1.509389	1.509389	1.590388
1.00	2.60	1.487957	1.486445	1.486372	1.486369	1.486369	1.486368
1.00	2.80	1.470044	1.468562	1.468490	1.468487	1.468487	1.468487
1.00	3.00	1.455869	1.454409	1.454339	1.454336	1.454336	1.454336
1.00	3.20	1.444469	1.443027	1.442958	1.442955	1.442955	1.442954
1.00	3.40	1.435169	1.433741	1.433673	1.433670	1.433670	1.433670
1.00	3.60	1.427487	1.426071	1.426004	1.426001	1.426001	1.426000
1.00	3.80	1.421070	1.419664	1.419598	1.419595	1.419595	1.419594
1.00	4.00	1.415657	1.414259	1.414193	1.414190	1.414190	1.414190
5.00	.60	6.079742	6.054852	6.053718	6.053785	6.053685	6.053683

β	γ	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7
5.00	.80	4.216763	4.193561	4.192492	4.192465	4.192465	4.192465
5.00	1.00	3.367707	3.346614	3.345654	3.345631	3.345631	3.345631
5.00	1.20	2.936885	2.918170	2.917347	2.917329	2.917329	2.917329
5.00	1.40	2.701752	2.685164	2.684464	2.684448	2.684448	2.684448
5.00	1.60	2.565764	2.550855	2.550247	2.550233	2.550233	2.550233
5.00	1.80	2.483231	2.469585	2.469043	2.469029	2.469029	2.469029
5.00	2.00	2.431068	2.418363	2.417867	2.417854	2.417854	2.417853
5.00	2.20	2.396950	2.384951	2.384487	2.384474	2.384474	2.384473
5.00	2.40	2.373971	2.362511	2.362071	2.362058	2.362058	2.362057
5.00	2.60	2.358102	2.347059	2.346636	2.346624	2.346623	2.346622
5.00	2.80	2.346903	2.336189	2.335780	2.335767	2.335766	2.335765
5.00	3.00	2.338849	2.328399	2.328000	2.327987	2.327987	2.327985
5.00	3.20	2.332963	2.322727	2.322335	2.322323	2.322322	2.322321
5.00	3.40	2.328598	2.318538	2.318153	2.318140	2.318140	2.318138
5.00	3.60	2.325322	2.315408	2.315029	2.315016	2.315015	2.315014
5.00	3.80	2.322836	2.313045	2.312670	2.312657	2.312656	2.312655
5.00	4.00	2.320937	2.311245	2.310874	2.310860	2.310860	2.310858

β	γ	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7
10.00	.60	7.407031	7.362011	7.360290	7.360254	7.360254	7.360253
10.00	.80	5.046014	5.008351	5.006938	5.006917	5.006917	5.006917
10.00	1.00	3.994874	3.962913	3.961745	3.961731	3.961730	3.961729
10.00	1.20	3.481179	3.454214	3.453273	3.453263	3.453262	3.453262
10.00	1.40	3.213533	3.190604	3.189842	3.189833	3.189833	3.189832
10.00	1.60	3.066861	3.046979	3.046343	3.046335	3.046335	3.046335
10.00	1.80	2.983167	2.965530	2.964978	2.964970	2.964970	2.964970
10.00	2.00	2.933878	2.917897	2.917403	2.917395	2.917395	2.917394
10.00	2.20	2.904166	2.889424	2.888969	2.888961	2.888961	2.888961
10.00	2.40	2.885979	2.872182	2.871756	2.871747	2.871747	2.871747
10.00	2.60	2.874774	2.861709	2.861303	2.861294	2.861294	2.861294
10.00	2.80	2.867895	2.855408	2.885018	2.855009	2.885009	2.855009
10.00	3.00	2.863748	2.851725	2.851347	2.851338	2.851337	2.851337
10.00	3.20	2.861349	2.849705	2.849336	2.849326	2.849326	2.849325
10.00	3.40	2.860081	2.848748	2.848387	2.848377	2.848377	2.848376
10.00	3.60	2.859547	2.848474	2.848119	2.848109	2.848109	2.848108
10.00	3.80	2.859489	2.848636	2.848285	2.848275	2.848275	2.848274

β	γ	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7
10.00	4.00	2.859738	2.849070	2.848724	2.848714	2.848714	2.848713
100.00	.60	10.590065	10.556769	10.556396	10.556389	10.556380	10.556374
100.00	.80	6.717184	6.686546	6.686174	6.686163	6.686150	6.686140
100.00	1.00	5.149633	5.123715	5.123392	5.123382	5.123369	5.123360
100.00	1.20	4.442670	4.422451	4.422220	4.422212	4.422202	4.422194
100.00	1.40	4.102324	4.087124	4.086974	4.086967	4.086961	4.086956
100.00	1.60	3.931809	3.920429	3.920332	3.920327	3.920322	3.920319
100.00	1.80	3.844771	3.836139	3.836073	3.836069	3.836066	3.836064
100.00	2.00	3.800600	3.793916	3.793869	3.793866	3.793864	3.793863
100.00	2.20	3.779148	3.773854	3.773818	3.773816	3.773815	3.773814
100.00	2.40	3.769986	3.765698	3.765669	3.765667	3.765666	3.765666
100.00	2.60	3.767526	3.763979	3.763955	3.763954	3.763953	3.763953
100.00	2.80	3.768696	3.765706	3.765685	3.765684	3.765684	3.765683
100.00	3.00	3.771772	3.769208	3.769189	3.769189	3.769188	3.769188
100.00	3.20	3.775776	3.773543	3.773525	3.773525	3.773525	3.773524
100.00	3.40	3.780149	3.778177	3.778161	3.778160	3.778160	3.778160
100.00	3.60	3.785474	3.782813	3.782797	3.782797	3.782797	3.782797
100.00	3.80	3.788878	3.787287	3.787272	3.787272	3.787272	3.787271
100.00	4.00	3.792968	3.791517	3.791502	3.791502	3.791502	3.791502

β	γ	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7
100000.00	.60	11.188913	11.167533	11.167462	11.167433	11.167433	11.167419
100000.00	.80	6.997676	6.973419	6.973267	6.973239	6.973222	6.973209
100000.00	1.00	5.333144	5.311530	5.311370	5.311347	5.311331	5.311319
100000.00	1.20	4.591832	4.575079	4.574969	4.574950	4.574938	4.574929
100000.00	1.40	4.238874	4.226717	4.226655	4.226639	4.226631	4.226625
100000.00	1.60	4.064236	4.055606	4.055574	4.055562	4.055556	4.055552
100000.00	1.80	3.976570	3.970440	3.970423	3.970415	3.970411	3.970408
100000.00	2.00	3.933197	3.928793	3.928785	3.928778	3.928775	3.928773
100000.00	2.20	3.913066	3.909854	3.909850	3.909845	3.909843	3.909841
100000.00	2.40	3.905350	3.902968	3.902965	3.902962	3.902960	3.902959
100000.00	2.60	3.904290	3.902493	3.902492	3.902489	3.902488	3.902487
100000.00	2.80	3.906744	3.905368	3.905367	3.905365	3.905364	3.905363
100000.00	3.00	3.910968	3.909899	3.909898	3.909896	3.909896	3.909895
100000.00	3.20	3.915985	3.915143	3.915143	3.915141	3.915141	3.915140
100000.00	3.40	3.921247	3.920576	3.920575	3.920574	3.920574	5.920574
100000.00	3.60	3.926452	3.925911	3.925910	3.925910	3.925909	3.925909
100000.00	3.80	3.931438	3.930997	3.930997	3.930997	3.930996	3.930996
100000.00	4.00	3.936127	3.935764	3.935764	3.935764	3.935764	3.935763

the solution rapidly converges to the so termed exact. This is demonstrated in Table 13.

The next case to be considered is that of biaxial compression with the load per unit length of the edge, constant and the restraint per unit length, equal on all faces. The two displacements corresponding to all sides simply supported and all sides clamped will be taken as the characteristic functions. In the former case the deflection function is

$$w = a_1 \sin m\pi\xi \ \sin\pi\eta$$

and in the latter case

$$w = a_2 \sin^2 m\pi\xi \ \sin^2\pi\eta$$

It is to be noted that the latter is not an exact extreme function, however, it closely approximates the true shape. The buckle mode shape is given by

$$w = \sum_{r=1}^{n} a_r \sin^r m\pi\xi \ \sin^r\pi\eta \tag{3.5}$$

When the normal algebraic manipulations, which arise from using this displacement function in an energy approach, have been completed, we arrive at the following expression.

$$m^4 \sum_r \left[a_1 \frac{1}{2} \cdot r\{(r-1)(r-2)(r-3)D_{3,M} - 2(r-1)(3r-4)D_{2,M} - (2-3r)D_{1,M}\}\right]$$

elastically restrained beams in x and y directions. The buckling coefficients obtained from this and the exact solutions (Chapter IV) for various values of the restraint parameter (β) are also indicated in Figure 9 for comparison. The close correspondence gives strong support to the proposed method of function selection for plate problems.

It is clear that, at the expense of little extra computer time, the open question of the stability of a plate with unequal but uniform edge restraint on four boundaries can be readily resolved. For example, consider a plate, as shown in Figure 10, under biaxial compression with boundary restraints, K_1, K_2, K_3 and K_4. If the restraint on each boundary is varied independently, we obtain sixteen combinations of extreme values of the restraint parameters. However, in this case,

$$\mu_{1,2,3,4} = \mu_{1,4,3,2} = \mu_{3,4,1,2} = \mu_{3,2,1,4}$$

where $\mu_{1,2,3,4}$ represents buckling coefficient corresponding to the restraint parameters β_1, β_2, β_3 and β_4. Hence the sixteen combinations reduce to 9 independent ones. These, and the corresponding assumed extreme functions are given in Figure 10.

The assumed general sequence of functions then becomes,

$$w = \sum_{r=o}^{m_1} \sum_{s=o}^{m_2} \cdots \sum_{j=o}^{m_9} C_{rs \cdots j} f_1^r f_2^s \cdots f_9^j \qquad (3.7)$$

Considering only the linear terms, i.e. the first approximation,

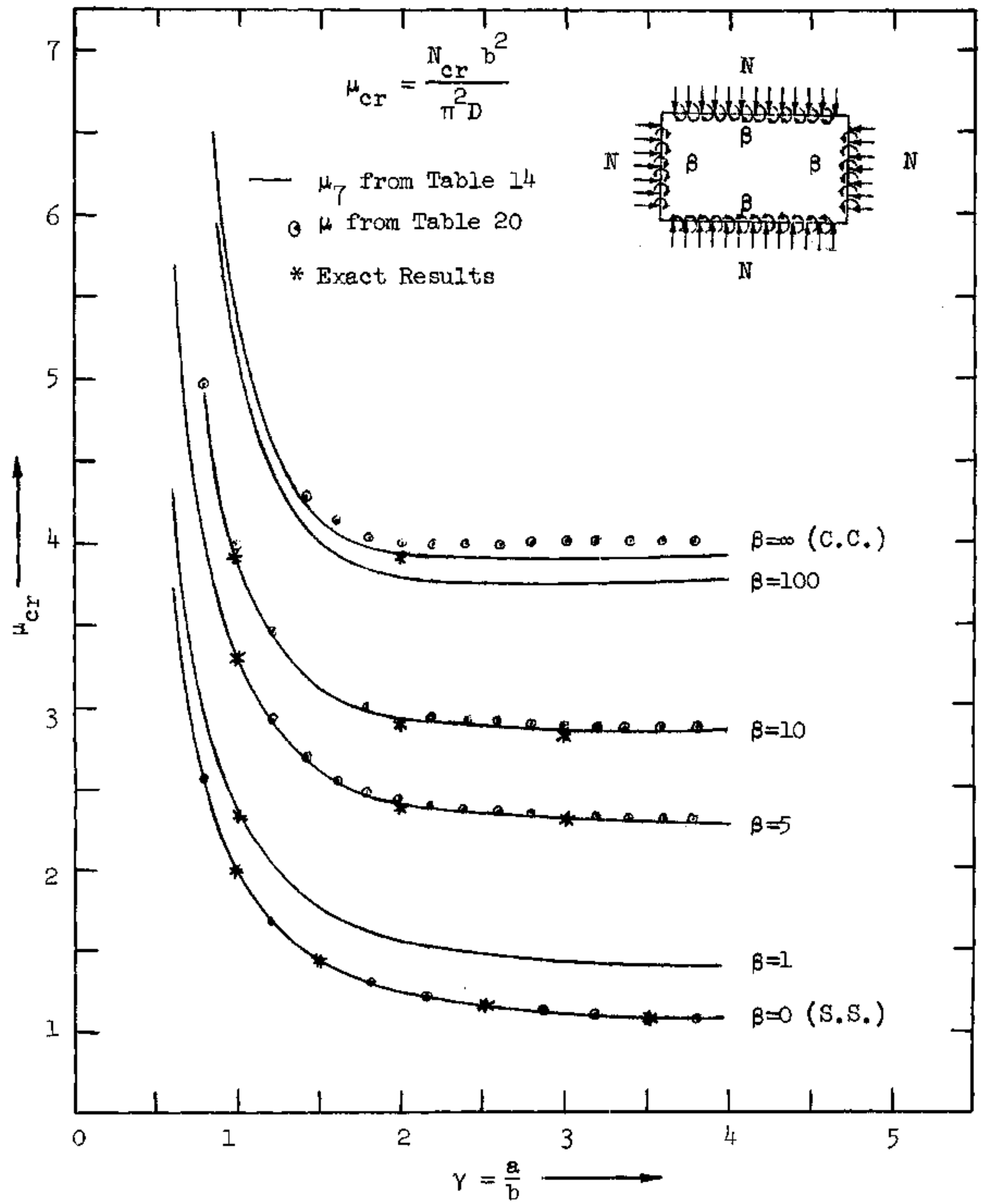
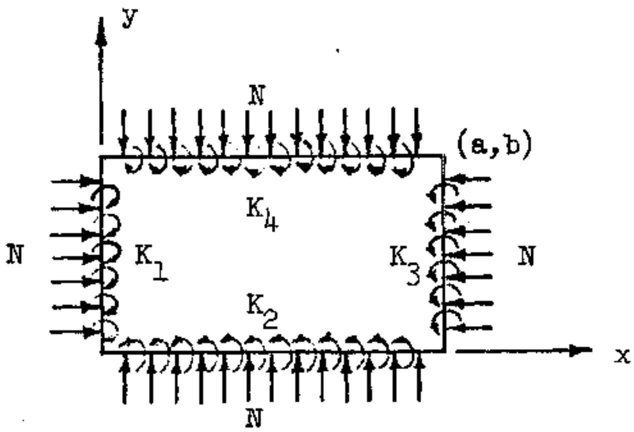


Figure 9. Rotationally Restrained Rectangular Plate Under Uniform Biaxial Compression. Buckling Coefficients by Various Methods.



$$\mu = \frac{Nb^2}{\pi^2 D}$$

$$\beta_i = \frac{K_i b}{D}$$

$(\beta_1, \beta_2, \beta_3, \beta_4)$	Assumed Function
$(0,0,0,0)$	$f_1 = \sin \pi\xi \ \sin \pi\eta$
$(0,0,0,\infty)$	$f_2 = \sin \pi\xi \left(\eta - \frac{\sin k\eta}{\sin k}\right)$
$(0,0,\infty,0)$	$f_3 = \left(\xi - \frac{\sin k\xi}{\sin k}\right) \sin \pi\eta$
$(0,0,\infty,\infty)$	$f_4 = \left(\xi - \frac{\sin k\xi}{\sin k}\right)\left(\eta - \frac{\sin k\eta}{\sin k}\right)$
$(0,\infty,0,\infty)$	$f_5 = \sin \pi\xi \ \sin^2 \pi\eta$
$(\infty,0,\infty,0)$	$f_6 = \sin^2 \pi\xi \ \sin \pi\eta$
$(0,\infty,\infty,\infty)$	$f_7 = \left(\xi - \frac{\sin k\xi}{\sin k}\right) \sin^2 \pi\eta$
$(\infty,0,\infty,\infty)$	$f_8 = \sin^2 \pi\xi \left(\eta - \frac{\sin k\eta}{\sin k}\right)$
$(\infty,\infty,\infty,\infty)$	$f_9 = \sin^2 \pi\xi \ \sin^2 \pi\eta$

Figure 10. Biaxially Compressed Rectangular Plate with Unequal Restraints on the Boundaries and its Independent Extreme Cases.

we get,

$$w = \sum_{r=1}^{9} a_r f_r \tag{3.8}$$

Using this function in the Rayleigh-Ritz process, we obtain a set of nine simultaneous homogeneous equations. The characteristic determinant of the coefficients is equated to zero for nontrivial solution and is solved numerically. The buckling coefficients, for aspect ratios of one and two and various values of restraint parameters are given in Table 15.*

It is pertinent, for the case of biaxially compressed plate with identical restraints, β, on all edges, to point out, that the strong analogy with the column problem should enable us to derive some very powerful and useful approximations. We note that to a first approximation, the deflected shape can be expressed as

$$w = \frac{2\pi \sin\pi\xi \sin\pi\eta + \beta \sin^2\pi\xi \sin^2\pi\eta}{2\pi + \beta} \tag{3.9}$$

i.e.

$$w = \frac{2\pi f_{0000} + \beta f_{\infty\infty\infty\infty}}{2\pi + \beta}$$

This deflection function is, of course, of similar form to

*Corresponding results obtained from the Rayleigh's Quotient, (Equation 3.17) are also given for comparison.

Table 15. Biaxially Compressed Rectangular Plate with Unequal Restraints on All Boundaries. Comparison of Buckling Coefficients by R.R. Method Using Extreme Functions, and Using Composite Function of True Beam Vibration Shapes.

β_1	β_2	β_3	β_4	$\gamma = 1$		$\gamma = 2$	
				(1)*	(2)*	(1)*	(2)*
0	0	0	0	2.0000	2.0000	1.2500	1.2500
0	0	0	5	2.3006	2.2967	1.7171	1.7047
0	0	0	10	2.4203	2.4141	1.8925	1.8697
0	0	0	10^5	2.6711	2.6649	2.2454	2.1790
0	0	5	0	2.3006	2.2967	1.3017	1.3016
0	0	5	5	2.5915	2.5862	1.7619	1.7531
0	0	5	10	2.7049	2.6988	1.9331	1.9165
0	0	5	10^5	2.9389	2.9349	2.2674	2.2217
0	0	10	0	2.4203	2.4141	1.3186	1.3178
0	0	10	5	2.7049	2.6988	1.7760	1.7668
0	0	10	10	2.8153	2.8093	1.9466	1.9293
0	0	10	10^5	3.0445	3.0397	2.2776	2.2331
0	0	10^5	0	2.6711	2.6649	1.3521	1.3438
0	0	10^5	5	2.9389	2.9349	1.8018	1.7840
0	0	10^5	10	3.0445	3.0397	1.9723	1.9440
0	0	10^5	10^5	3.2653	3.2587	2.3052	2.2451
0	5	0	5	2.6843	2.6755	2.3100	2.3074
0	5	0	10	2.8395	2.8297	2.5342	2.5346
0	5	0	10^5	3.1674	3.1609	2.9806	2.9598
0	5	5	5	2.9680	2.9601	2.3481	2.3526
0	5	5	10	3.1174	3.1101	2.5692	2.5781

*(1) Rayleigh-Ritz method using extreme functions (1st Approximation).

(2) Rayleigh's Quotient using true vibration mode shape of a beam (Eqn. 3-17).

Table 15. Biaxially Compressed Rectangular Plate with Unequal Restraints on All Boundaries. Comparison of Buckling Coefficients by R.R. Method Using Extreme Functions, and Using Composite Function of True Beam Vibration Shapes. (Continued)

β_1	β_2	β_3	β_4	$\gamma = 1$ (1)	$\gamma = 1$ (2)	$\gamma = 2$ (1)	$\gamma = 2$ (2)
0	0	5	10^5	3.4301	3.4270	3.0065	2.9992
0	5	10	5	3.0742	3.0677	2.3568	2.3629
0	5	10	10	3.2203	3.2152	2.5768	2.5873
0	5	10	10^5	3.5272	3.5258	3.0158	3.0065
0	5	10^5	5	3.2871	3.2853	2.3699	2.3677
0	5	10^5	10	3.4256	3.4255	2.5882	2.5881
0	5	10^5	10^5	3.7190	3.7217	3.0369	3.0030
0	10	0	10	3.0159	3.0012	2.7841	2.7909
0	10	0	10^5	3.3833	3.3727	3.2728	3.2753
0	10	5	10	3.2880	3.2778	2.8141	2.8328
0	10	5	10^5	3.6400	3.6352	3.2992	3.3129
0	10	10	10	3.3873	3.3804	2.8194	2.8406
0	10	10	10^5	3.7325	3.7311	3.3071	3.3188
0	10	10^5	10	3.5834	3.5825	2.8272	2.8369
0	10	10^5	10^5	3.9136	3.9173	3.3228	3.3097
0	10^5	0	10^5	3.8571	3.8455	3.8360	3.8971
0	10^5	5	10^5	4.0949	4.0937	3.8429	3.9304
0	10^5	10	10^5	4.1790	4.1824	3.8433	3.9339
0	10^5	10^5	10^5	4.3808	4.3493	3.8441	3.9177
5	0	5	0	2.6843	2.6755	1.3648	1.3679
5	0	5	5	2.9680	2.9601	1.8157	1.8180
5	0	5	10	3.0742	3.0677	1.9777	1.9800
5	0	5	10^5	3.2871	3.2853	2.2858	2.2804
5	0	10	0	2.8395	2.8297	1.3855	1.3893
5	0	10	5	3.1174	3.1101	1.8326	1.8376
5	0	10	10	3.2203	3.2152	1.9926	1.9988
5	0	10	10^5	3.4256	3.4255	2.2949	2.2974

Table 15. Biaxially Compressed Rectangular Plate with Unequal Restraints on All Boundaries. Comparison of Buckling Coefficients by R.R. Method Using Extreme Functions, and Using Composite Function of True Beam Vibration Shapes. (Continued)

β_1	β_2	β_3	β_4	$\gamma = 1$ (1)	$\gamma = 1$ (2)	$\gamma = 2$ (1)	$\gamma = 2$ (2)
5	0	10^5	0	3.1674	3.1609	1.4244	1.4233
5	0	10^5	5	3.4301	3.4270	1.8640	1.8636
5	0	10^5	10	3.5272	3.5258	2.0221	2.0223
5	0	10^5	10^5	3.7190	3.7217	2.3212	2.3178
5	5	5	5	3.3448	3.3347	2.4069	2.4178
5	5	5	10	3.4895	3.4814	2.6239	2.6425
5	5	5	10^5	3.7835	3.7816	3.0306	3.0591
5	5	10	5	3.4895	3.4814	2.4210	2.4352
5	5	10	10	3.6312	3.6259	2.6365	2.6589
5	5	10	10^5	3.9182	3.9188	3.0397	3.0737
5	5	10^5	5	3.7835	3.7816	2.4434	2.4500
5	5	10^5	10	3.9182	3.9188	2.6565	2.6702
5	5	10^5	10^5	4.1875	4.1950	3.0586	3.0804
5	10	5	10	3.6561	3.6466	2.8706	2.8964
5	10	5	10^5	3.9908	3.9883	3.3428	3.3725
5	10	10	10	3.7946	3.7889	2.8820	2.9120
5	10	10	10^5	4.1215	4.1232	3.3479	3.3860
5	10	10^5	10	4.0731	4.0739	2.8990	2.9190
5	10	10^5	10^5	4.3811	4.3903	3.3508	3.3877
5	10^5	5	10^5	4.4258	4.4324	3.8092	3.9864
5	10^5	10	10^5	4.5475	4.5597	3.8998	3.9977
5	10^5	10^5	10^5	4.7858	4.8062	3.9080	3.9926
10	10	10	10	3.9512	3.9498	2.8953	2.9302
10	0	10	0	3.0159	3.0012	1.4084	1.4125
10	0	10	5	3.2880	3.2779	1.8592	1.8592
10	0	10	10	3.3873	3.3804	2.0084	2.0196
10	0	10	10^5	3.5841	3.5825	2.3032	2.3165

Table 15. Biaxially Compressed Rectangular Plate with Unequal Restraints on All Boundaries. Comparison of Buckling Coefficients by R.R. Method Using Extreme Functions, and Using Composite Function of True Beam Vibration Shapes. (Continued)

β_1	β_2	β_3	β_4	$\gamma = 1$ (1)	$\gamma = 1$ (2)	$\gamma = 2$ (1)	$\gamma = 2$ (2)
10	0	10^5	0	3.3845	3.3727	1.4499	1.4486
10	0	10^5	5	3.6405	3.6352	1.8842	1.8876
10	0	10^5	10	3.7329	3.7311	2.0385	2.0457
10	0	10^5	10^5	3.9136	3.9173	2.3274	2.3393
10	5	10	5	3.6561	3.6466	2.4372	2.4551
10	5	10	10	3.7946	3.7889	2.6509	2.6781
10	5	10	10^5	4.0730	4.0739	3.0484	3.0909
10	5	10^5	5	3.9904	3.9883	2.4625	2.4729
10	5	10^5	10	4.1211	4.1232	2.6734	2.6924
10	5	10^5	10^5	4.3809	4.3903	3.0153	3.1007
10	10	10	10	3.9512	3.9498	2.8953	2.9302
10	10	10	10^5	4.2732	4.2763	3.3441	3.4023
10	10	10^5	10	4.2732	4.2763	2.9150	2.9405
10	10	10^5	10^5	4.5713	4.5840	3.3597	3.4074
10	10^5	10	10^5	4.6894	4.7050	3.9091	4.0120
10	10^5	10^5	10^5	4.9652	4.9913	3.9183	4.0103
10^5	0	10^5	0	3.8571	3.8455	1.5000	1.4833
10^5	0	10^5	5	4.0949	4.0937	1.9222	1.9148
10^5	0	10^5	10	4.1750	4.1824	2.0707	2.0706
10^5	0	10^5	10^5	4.3408	4.3493	2.3458	2.3612
10^5	5	10^5	5	4.4258	4.4324	2.4932	2.4898
10^5	5	10^5	10	4.5475	4.5597	2.7004	2.7059
10^5	5	10^5	10^5	4.7858	4.8062	3.0844	3.1099
10^5	10	10^5	10	4.6894	4.7050	2.9389	2.9502
10^5	5	10^5	10^5	4.9652	4.9913	3.3763	3.4119
10^5	10^5	10^5	10^5	5.3331	5.3750	3.9318	4.0083

that developed for a restrained column. We recollect that, for this latter structure, the critical load, to a first approximation, could be written as

$$P_{cr} = \frac{\beta \, P_{\infty\infty} + 2\pi \, P_{OO}}{\beta + 2\pi}$$

and we therefore conjecture, that for the plate, a similar situation might exist, viz.

$$\mu_{cr} \cong \frac{\beta \, \mu_{\infty\infty\infty\infty} + 2\pi \, \mu_{OOOO}}{\beta + 2\pi} \tag{3.10}$$

This, in fact, is true as can be seen from the results given in Table 16. The accuracy is within 3.5%.

The uniaxially compressed plate is more difficult to deal with. This is due to the fact that for uniaxial loading, the critical condition is dependent upon the number of half waves in the longitudinal direction as well as the plate aspect ratio and the boundary conditions. It is the former condition which causes the curves of buckling coefficient versus aspect ratio to take on their wellknown scalloped form. It is significant to note that the minimum coefficients corresponding to each specific wave number and the intersection points of consecutive wave number curves lie on smooth curves whose equations are simply and readily definable as a function of the restraint parameter. The reason for the shift in the minimum buckling coefficients is that as the boundary restraint increases, the inflexion

Table 16. Comparison of Buckling Coefficients Given in Table 14 and Those Obtained From Equation (3-10).

β	γ	$\mu_{7(R.R.)}$	μ(Equation 3-10)	% Discrepancy
.00	1.00	2.000000	2.000000	.0000
.00	1.20	1.694444	1.694444	.0000
.00	1.40	1.510204	1.510204	.0000
.00	1.60	1.390625	1.390625	.0000
.00	1.80	1.308642	1.308642	.0000
.00	2.00	1.250000	1.250000	.0000
.00	2.20	1.206611	1.206612	.0000
.00	2.40	1.173611	1.173611	.0000
.00	2.60	1.147929	1.147929	.0000
.00	2.80	1.127551	1.127551	.0000
.00	3.00	1.111111	1.111111	.0000
.00	3.20	1.097656	1.097656	.0000
.00	3.40	1.086505	1.086505	.0000
.00	3.60	1.077160	1.077160	.0000
.00	3.80	1.069252	1.069252	.0000
.00	4.00	1.062500	1.062500	.0000
.10	1.00	2.040158	2.051875	.5743

Table 16. Comparison of Buckling Coefficients Given in Table 14 and Those Obtained From Equation (3-10). (Continued)

β	γ	μ_7(R.R.)	μ(Equation 3-10)	% Discrepancy
.10	1.20	1.731836	1.739555	.4457
.10	1.40	1.546455	1.552721	.4052
.10	1.60	1.426520	1.432318	.4065
.10	1.80	1.344558	1.350277	.4254
.10	2.00	1.286111	1.291899	.4501
.10	2.20	1.242989	1.248895	.4752
.10	2.40	1.210275	1.216307	.4984
.10	2.60	1.184874	1.191024	.5190
.10	2.80	1.164761	1.171014	.5369
.10	3.00	1.148565	1.154907	.5522
.10	3.20	1.135331	1.141749	.5653
.10	3.40	1.124380	1.130862	.5765
.10	3.60	1.115215	1.121751	.5861
.10	3.80	1.107468	1.114049	.5943
.10	4.00	1.100861	1.107480	.6013
1.00	1.00	2.370419	2.454650	3.5534
1.00	1.20	2.037459	2.089805	2.5692

Table 16. Comparison of Buckling Coefficients Given in Table 14 and Those Obtained From Equation (3-10). (Continued)

β	γ	$\mu_{7(R.R.)}$	μ(Equation 3-10)	% Discrepancy
1.00	1.40	1.842043	1.882833	2.2144
1.00	1.60	1.719032	1.756038	2.1527
1.00	1.80	1.637274	1.673547	2.2155
1.00	2.00	1.580511	1.617219	2.3225
1.00	2.20	1.539667	1.577198	2.4376
1.00	2.40	1.509388	1.547810	2.5456
1.00	2.60	1.486368	1.525624	2.6411
1.00	2.80	1.468487	1.508477	2.7232
1.00	3.00	1.454336	1.494953	2.7929
1.00	3.20	1.442954	1.484102	2.8516
1.00	3.40	1.433670	1.475261	1.9010
1.00	3.60	1.426000	1.467962	2.9427
1.00	3.80	1.419594	1.461866	2.9778
1.00	4.00	1.414190	1.456721	3.0075
5.00	1.00	3.345631	3.467361	3.6385
5.00	1.20	2.917329	2.970450	1.8209
5.00	1.40	2.684448	2.712845	1.0578

Table 16. Comparison of Buckling Coefficients Given in Table 14 and Those Obtained From Equation (3-10). (Continued)

β	γ	$\mu_{7(R.R.)}$	$\mu_{(Equation\ 3\text{-}10)}$	% Discrepancy
5.00	1.60	2.550233	2.569977	.7742
5.00	1.80	2.469029	2.486356	.7018
5.00	2.00	2.417853	2.435180	.7166
5.00	2.20	2.384473	2.402660	.7627
5.00	2.40	2.362057	2.381320	.8155
5.00	2.60	2.346622	2.366922	.8651
5.00	2.80	2.335765	2.356969	.9078
5.00	3.00	2.327985	2.349943	.9432
5.00	3.20	2.322321	2.344890	.9718
5.00	3.40	2.318138	2.341195	.9946
5.00	3.60	2.315014	2.338454	1.0125
5.00	3.80	2.312655	2.336394	1.0265
5.00	4.00	2.310858	2.334828	1.0373
10.00	1.00	3.961729	4.033571	1.8134
10.00	1.20	3.453262	3.462822	.2768
10.00	1.40	3.189832	3.176907	-.4052
10.00	1.60	3.046335	3.025053	-.6986
10.00	1.80	2.964970	2.940799	-.8152

Table 16. Comparison of Buckling Coefficients Given in Table 14 and Those Obtained From Equation (3-10). (Continued)

β	γ	μ_7(R.R.)	μ(Equation 3-10)	% Discrepancy
10.00	2.20	2.888961	2.864179	-.8578
10.00	2.40	2.871747	2.847338	-.8500
10.00	2.60	2.861294	2.837294	-.8388
10.00	2.80	2.855009	2.831364	-.8282
10.00	3.00	2.851337	2.827971	-.8195
10.00	3.20	2.849325	2.826159	-.8130
10.00	3.40	2.848376	2.825342	-.8087
10.00	3.60	2.848108	2.825149	-.8061
10.00	3.80	2.848274	2.825346	-.8050
10.00	4.00	2.848713	2.825781	-.8050
100.00	1.00	5.123360	5.115545	-.1525
100.00	1.20	4.422194	4.403698	-.4183
100.00	1.40	4.086956	4.063687	-.5693
100.00	1.60	3.920319	3.894661	-.6545
100.00	1.80	3.836064	3.809199	-.7003
100.00	2.00	3.793863	3.766409	-.7236
100.00	2.20	3.773814	3.746098	-.7344
100.00	2.40	3.765666	3.373855	-.7385

Table 16. Comparison of Buckling Coefficients Given in Table 14 and Those Obtained From Equation (3-10). (Continued)

β	γ	$\mu_{7(R.R.)}$	μ(Equation 3-10)	% Discrepancy
100.00	2.60	3.763953	3.736131	-.7392
100.00	2.80	3.765683	3.737889	-.7381
100.00	3.00	3.769188	3.741436	-.7363
100.00	3.20	3.773524	3.745820	-.7342
100.00	3.40	3.778160	3.750500	-.7321
100.00	3.60	3.782797	3.755177	-.7301
100.00	3.80	3.787271	3.759686	-.7284
100.00	4.00	3.791502	3.763946	-.7268
100000.00	1.00	5.311319	5.311092	-.0043
100000.00	1.20	4.574929	4.573744	-.0259
100000.00	1.40	4.226625	4.223957	-.0631
100000.00	1.60	4.055552	4.051827	-.0919
100000.00	1.80	3.970408	3.966146	-.1073
100000.00	2.00	3.928773	3.924352	-.1125
100000.00	2.20	3.909841	3.905489	-.1113
100000.00	2.40	3.902959	3.898800	-.1066
100000.00	2.60	3.902487	3.898579	-.1001
100000.00	2.80	3.905363	3.901727	-.0931

Table 16. Comparison of Buckling Coefficients Given in Table 14 and Those Obtained From Equation (3-10). (Continued)

β	γ	$\mu_{7(R.R.)}$	μ(Equation 3-10)	% Discrepancy
100000.00	3.00	3.909895	3.906529	-.0861
100000.00	3.20	3.915140	3.912032	-.0794
100000.00	3.40	3.920574	3.917706	-.0731
100000.00	3.60	3.925909	3.923263	-.0674
100000.00	3.80	3.930996	3.928551	-.0622
100000.00	4.00	3.935763	3.933502	-.0575

point contour moves away from the edges of the plate (see Figure 11). It is interesting to note that the buckling coefficient curve for any specific restraint parameter can be obtained from that corresponding to the simply supported conditions by appropriate adjustment of the aspect ratio and a transformation function of the restraint parameter. Hence the buckling coefficient for a particular β and γ is given by

$$\mu_{(\beta,\gamma)} \cong \mu_o^* \; f(\beta) \qquad (3.11)$$

where

μ^* is the buckling coefficient for simply supported conditions for an apparent aspect ratio γ^* and $f(\beta)$ is the critical load transformation function.

The "Apparent aspect ratio", γ^* is defined as the aspect ratio of the corresponding simply supported plate, obtained from the equation of the locus of minimum coefficients. For the case under consideration, Equation (3.11) can be written as

$$\mu_{(\beta,\gamma)} \cong \left\{ \left(\frac{m}{\gamma^*}\right) + \left(\frac{\gamma^*}{m}\right)\right\}^2 \left(\frac{1.75\beta + 8.86}{\beta + 8.86}\right) \qquad (3.12)$$

where

$$\gamma^* \cong \left(\frac{3\beta + 10}{2\beta + 10}\right)\gamma$$

In Equation (3.12), the apparent aspect ratio, γ^*, and the

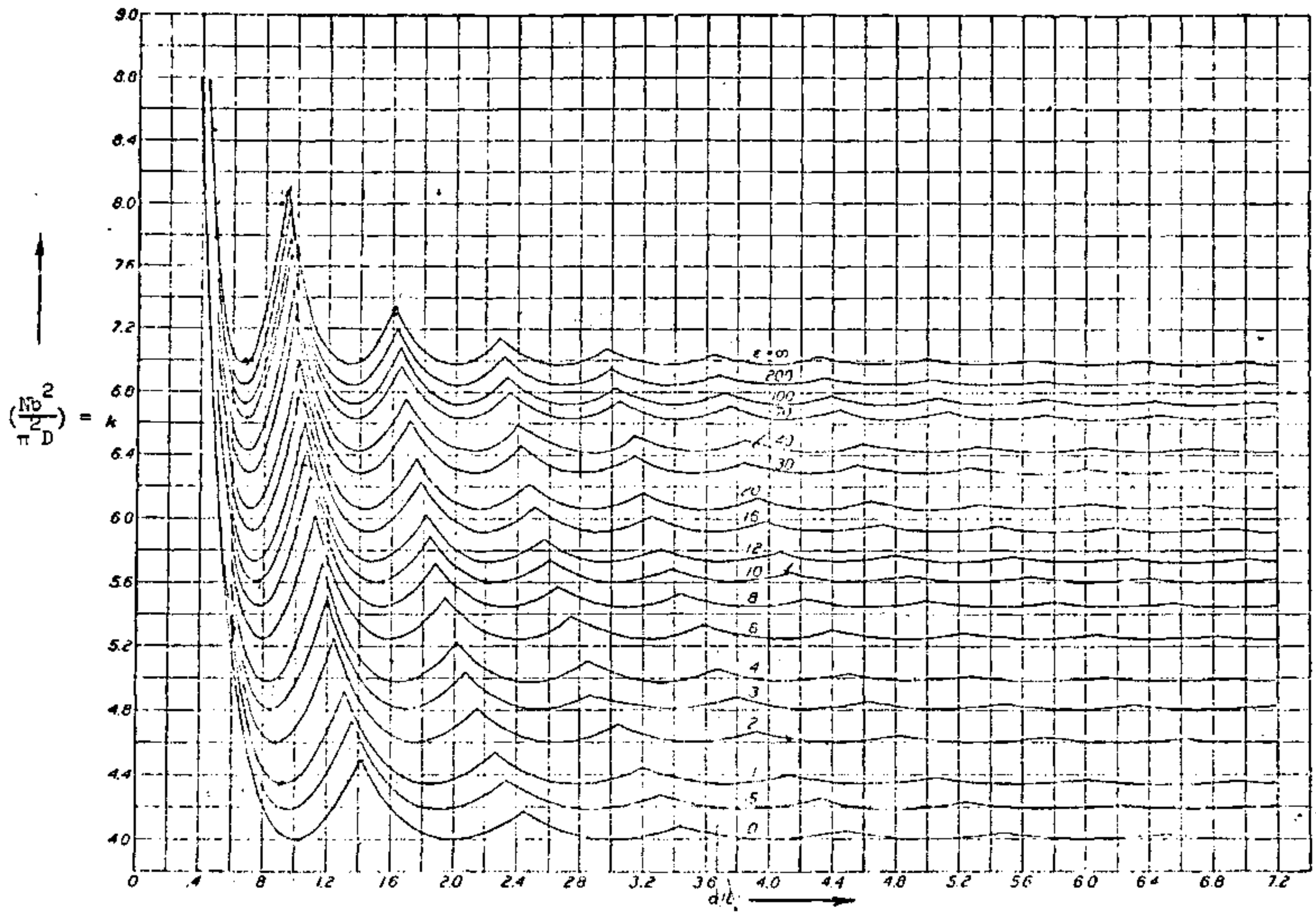


Figure 11. Buckling Coefficient Curves for a Uniaxially Compressed Rectangular Plate with Simply Supported Loaded Edges and Rotationally Restrained Unloaded Edges from Reference [20].

transformation function are obtained from the locus of minimum buckling coefficients in Figure 11.

The accuracy of this expression is very good, as seen from Table 17, where the results for (m/γ) values between intersection points of consecutive wave number curves are given. For this range of $\left(\frac{m}{\gamma}\right)$, the error is less than 3 percent. This suggests a possibility of tackling more complex problems involving similar scallop curves with ease and generality.

Another simple and useful approximation can be developed for the square plate under biaxial compression and with different uniform restraints on the boundary. Let us define $\mu_{1,2,3,4}$ as the buckling coefficient corresponding to the restraint parameters β_1, β_2, β_3 and β_4 and μ_i as the coefficient corresponding to β_i, β_i, β_i, β_i.

For this case, we have

$$\mu_{rsjt} = \sqrt{\mu_{rsjj}\ \mu_{rstt}}$$

and

$$\mu_{rsjt} = \frac{1}{2}\left(\mu_{rsjs} + \mu_{rtjt}\right)$$

Similar relationships exist for column and plate problems (19, 20, 21, 22, 23). Thus when all the restraints are varied independently, the buckling coefficient is given by

$$\mu_{1,2,3,4} \cong \frac{1}{2}\ \sqrt[4]{(\mu_1 + \mu_2)(\mu_2 + \mu_3)(\mu_3 + \mu_4)(\mu_4 + \mu_1)} \qquad (3.13)$$

Table 17. Accuracy of the Expression (3.12).

β	$\left(\frac{m}{\gamma}\right)$	μ_6 (Table 13)	μ (Equation 3.12)	% Discrepancy
0	1/.6	5.1377	5.1377	0.0000
0	1/.8	4.2025	4.2025	0.0000
0	1/1.0	4.0000	4.0000	0.0000
0	1/1.2	4.1344	4.1344	0.0000
0	1/1.4	4.4702	4.4702	0.0000
1	1/.6	5.2735	5.1537	2.2700
1	1/.8	4.4422	4.3930	1.1074
1	1/1.0	4.3731	4.3319	0.9400
1	1/1.2	4.6705	4.6074	1.3509
1	1/1.4	5.1987	5.0952	1.9914
5	1/.6	5.6682	5.5146	2.7099
5	1/.8	5.1211	5.0823	0.7586
5	1/1.0	5.4142	5.3395	1.3790
5	1/1.2	6.1525	5.9646	3.0307
10	1/.6	5.9702	5.8737	1.6164
10	1/.8	5.6227	5.6140	0.1550
10	1/1.0	6.1682	6.0663	1.6526
∞	1/.6	7.0550	7.0777	-0.3225
∞	1/.8	7.3033	7.2350	0.9357

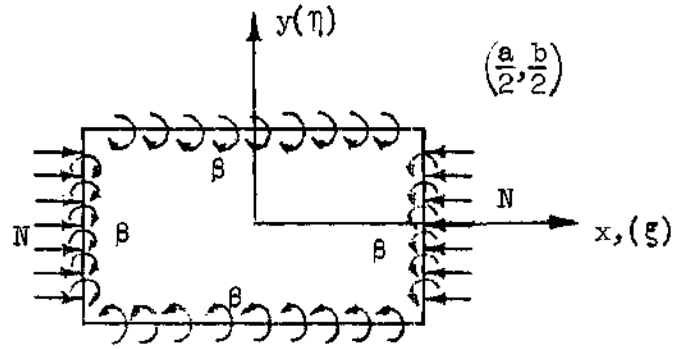
This is a very convenient and simple relationship. The accuracy of this expression is demonstrated in Table 18. Maximum error is seen to be 6 percent in the extreme case (o, o, ∞, ∞). Next an uniaxial loaded rectangular plate with equal restraints around its boundary is considered. It is clear that no matter to what degree the approximation is taken, the greater accuracy for a general situation is obtained when the characteristic functions are exact. Each deviation from exactness implies a loss in accuracy and the total loss must clearly be a function of the discrepancies in the individual characteristic function. Nevertheless, good accuracy is obtained if the approximate extreme functions, admissible in the sense of calculus of variations, are chosen so that they vary closely approximate the true functions. This was the reason for the outstanding accuracy obtained in the previous cases.

In the present case, the function $\sin^2 m\pi\xi$ poorly approximates the true shape for the clamped plate for $m > 1$, where m represents the number of half waves in the longitudinal direction. However, it has been shown in reference [24] that if we take the deflection shape to be a composite of the vibration mode shapes of clamped beams, it closely approximates the true buckle shape.* Moreover good accuracy is expected since these are the normal modes. Thus, the first approximation to the uniaxially compressed plate with identical boundary restraints is taken to be (Figure 12)

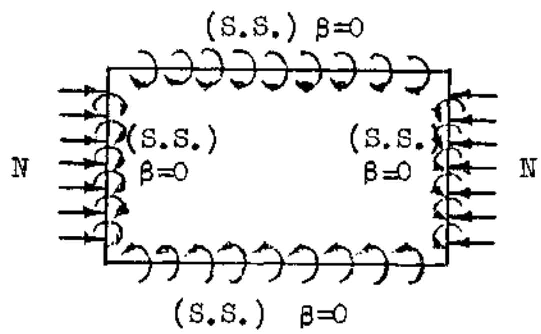
*Ashton and Waddoup have applied the concept in a broader sense [25].

Table 18. Biaxially Compressed Square Plate with Unequal Restraints, Comparison of Buckling Coefficients by R.R. Method (Table 15) and Those Obtained From Equation (3.13).

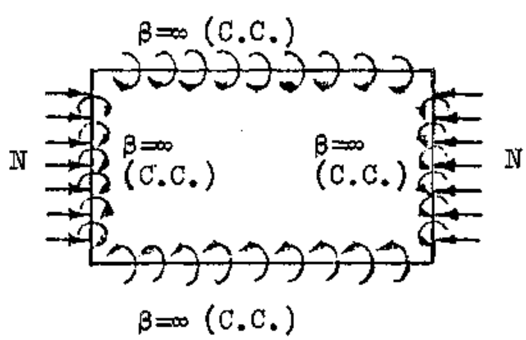
β_1	β_2	β_3	β_4	$\mu_{R.R.}$	$\mu_{1,2,3,4}$	% Discrepancy
0	0	0	0	2.0000	2.0000	0.0000
0	0	0	10	2.4181	2.4484	1.2531
0	0	0	∞	2.6711	2.7080	1.3815
0	0	10	10	2.8125	2.9108	3.4951
0	0	10	∞	3.0445	3.1820	4.5164
0	0	∞	∞	3.2653	3.4605	5.9781
0	10	0	10	3.0126	2.9974	-0.5046
0	10	0	∞	3.3833	3.3152	-2.0128
0	10	10	10	3.3841	3.4604	2.2547
0	10	10	∞	3.7325	3.7829	1.3503
0	∞	0	∞	3.8571	3.6667	-4.9364
0	∞	10	∞	4.1790	4.1354	-1.0433
0	∞	∞	∞	4.3408	4.4222	1.8752
10	0	10	∞	3.5834	3.7390	4.3423
10	10	10	∞	4.2732	4.3165	1.0133
10	10	∞	∞	4.5713	4.6399	1.5000
10	0	∞	∞	3.9136	4.0663	3.9018
10	∞	10	∞	4.6894	4.6641	-0.5395
10	∞	∞	∞	4.9652	4.9875	0.4491
∞	∞	∞	∞	5.3333	5.3333	0.0000



$$w = \frac{a_1 f_1 + a_2 f_2}{a_1 + a_2}$$



$$w = a_1 f_1 = a_1 \cos n\pi\xi \cos \pi\eta$$



$$w = a_2 f_2 = a_2\left(\cos \frac{k_m}{2} \cosh k_m\xi - \cosh \frac{k_m}{2} \cos k_m\xi\right)$$
$$\left(\cos \frac{p}{2} \cosh p\eta - \cosh \frac{p}{2} \cos p\eta\right)$$

Figure 12. Uniaxially Compressed Rectangular Plate with Equal Boundary Restraints and its Extreme Cases.

$$w = a_1 \cos n\pi\xi \cos \pi\eta$$

$$+ a_2(c_m \cosh k_m\xi - c_{mh} \cos k_m\xi)(D \cosh p\eta - Dh \cos p\eta) \tag{3.14}$$

where

m is odd,

$c_m = \cos (k_{m/2})$

$c_{mh} = \cosh (k_{m/2})$

$D = \cos (p/2)$

$D_h = \cosh (p/2)$

$k_1 = 4.73$

$k_3 = 10.9954$

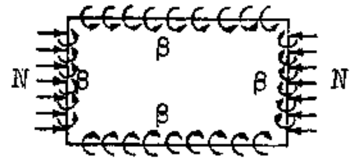
$p = 4.73$

n = number of half waves along the length for simply supported condition

Using this function in the Rayleigh-Ritz process, we obtain a quadratic expression (see Appendix B). Results of this expression are given in Table 19 for odd number of half waves along the length, along with the exact results for comparison, and a close agreement is seen. Similar analysis can be made for even number of half waves in clamped plate function.

The Rayleigh Quotient can be obtained for this case by using

Table 19. Rotationally Restrained Uniaxially Compressed Rectangular Plate. Buckling Coefficients by Rayleigh-Ritz Method and Exact Solutions.



m	γ	β = 0 (S.S. Plate)			m	γ	β = 1		
		μ_1	μ_2	μ_3			μ_1	μ_2	μ_3
1	.6	5.1378			1	.6	5.9198	5.9116	
1	.8	4.2025			1	.8	4.9101	4.9115	
1	1.0	4.0000			1	1.0	4.7385	4.7414	4.73
1	1.2	4.1344			1	1.2	4.9692	4.9720	
1	1.4	4.4702			1	1.4	5.4503	5.453	
2	1.4	4.5308			2	1.2	5.5854		
2	1.6	4.2025			2	1.4	4.9802		
2	1.8	4.0445			2	1.6	4.6729	4.7150	
2	2.0	4.0000			2	1.8	4.5513	4.5979	
2	2.2	4.0360			2	2.0	4.5558	4.6066	
2	2.4	4.1344			2	2.2	4.6527	4.7073	
3	2.2	4.3973			3	2.0	5.0512	5.0740	
3	2.4	4.2025	SAME AS μ_1	SAME AS μ_2	3	2.2	4.7710	4.7829	
3	2.6	4.0825			3	2.4	4.5995	4.5950	
3	2.8	4.0191			3	2.6	4.5083		
3	3.0	4.0000			3	2.8	4.4788		
3	3.2	4.0300			3	3.0	4.4983		
3	3.4	4.0630			3	3.2	4.5581		

μ_1 Critical load by R. R. Method; using a composite function of true beam vibration shapes; Rayleigh's Quotient, Equation 3.16

μ_2 Critical load by R. R. Method, using linear combination of extreme functions (vibration mode shapes of beam)

μ_3 Exact solution (Chapter IV)

Table 19. Rotationally Restrained Uniaxially Compressed Rectangular Plate. Buckling Coefficients by Rayleigh-Ritz Method and Exact Solutions. (Continued)

m	γ	β = 5			m	γ	β = ∞ (Clamped Plate)		
		μ_1	μ_2	μ_3			μ_1	μ_2	μ_3*
1	.6	8.2964	8.2300		1	0.75	12.10	12.10	11.66
1	.8	6.8557	6.8890	6.6200	1	1.00	10.75	10.75	10.07
1	1.0	6.6693	6.6965	6.5998					
1	1.2	7.1047	7.1244	6.8900	2	1.25	9.50		9.25
					2	1.50	8.70		8.33
2	1.0	8.0921			2	1.75	8.58		8.11
2	1.2	6.8379							
2	1.4	6.2103			3	2	8.29		7.88
2	1.6	5.9433				2.25	8.05		7.63
2	1.8	5.9085				2.50	8.05		7.57
2	2.0	6.0374							
2	2.2	6.2906			4	2.75	7.85		7.44
						3.00	7.78		7.37
3	2.0	6.0748		6.0363					
3	2.2	5.8327		5.7900					
3	2.4	5.7181		5.6000					
3	2.6	5.7001		5.5824					
3	2.8	5.7582							
3	3.0	5.8788							
3	3.2	6.0520							

*Results from reference [26]

the composite function formed as a product of exact vibration mode shapes of elastically restrained beams,

i.e.

$$w = A\ \varphi(\xi)\ \psi(\eta) \tag{3.15}$$

where

$$\varphi(\xi) = C_1 \cosh k_m\xi + C_2 \sinh k_m\xi + C_3 \cos k_m\xi + C_4 \sin k_m\xi$$

$$\psi(\eta) = D_1 \cosh p\eta + D_2 \sinh p\eta + D_3 \cos p\eta + D_4 \sin p\eta$$

$$C_1 = \sin \frac{k_m}{2} \cos \frac{k_m}{2}$$

$$C_2 = -\frac{1}{2} \sin k_m \left\{ \frac{(\beta_3-\beta_1)\left[\sinh \frac{k_m}{2} + \cosh \frac{k_m}{2} \sin \frac{k_m}{2} / \cos \frac{k_m}{2}\right]}{4k_m \sinh \frac{k_m}{2} + (\beta_1+\beta_3)\left[\cosh \frac{k_m}{2} - \sinh \frac{k_m}{2} \cot \frac{k_m}{2}\right]} \right\}$$

$$= -\frac{1}{2} \sin k_m\ \theta_o$$

$$C_3 = -\sin \frac{k_m}{2} \cosh \frac{k_m}{2}$$

$$C_4 = \cos \frac{k_m}{2} \sinh \frac{k_m}{2}\ \theta_o$$

D_1, D_2, D_3, D_4 are obtained by replacing k_m, β_1, β_3 by p, β_2 and β_4 respectively. Above coefficients C_1 through D_4 are for odd number of half waves along the length. For even number of half waves these constants are given by

$$C_1 = -\frac{1}{2}\sin k_m\,\theta_e$$

$$C_2 = \frac{1}{2}\sin k_m$$

$$C_3 = \sin\frac{k_m}{2}\cosh\frac{k_m}{2}\,\theta_e$$

$$C_4 = -\cos\frac{k_m}{2}\sinh\frac{k_m}{2}$$

$$\theta_e = \frac{(\beta_3-\beta_1)\left(\cosh\frac{k_m}{2} - \sinh\frac{k_m}{2}/\tan\frac{k_m}{2}\right)}{4k_m\cosh\frac{k_m}{2} + (\beta_1+\beta_3)\left(\sinh\frac{k_m}{2} + \cosh\frac{k_m}{2}\cdot\tan\frac{k_m}{2}\right)}$$

D_1, D_2, D_3 and D_4 are obtained by replacing k_m, β_1, β_3 by p, β_2 and β_4 respectively. The frequency parameters k_m and p for the clamped beam are given by

$$k_1 = 4.732 = p$$

$$k_2 = 7.8531$$

$$k_3 = 10.9954$$

$$k_4 = 14.1369$$

Using Equation (3.15), Rayleigh's Quotient is obtained which is given by

$$\mu = \frac{R_2R_3k_m^4 + 2p^2k_m^2\gamma^2R_5R_6 + p^4\gamma^4R_1R_4 + S}{\gamma^2k_m^2R_2R_9} \tag{3.16}$$

where

$$S = \gamma k_m^2 R_2(\beta_1 V_1 + \beta_3 V_3) + \gamma^4 p^2 R_1(V_2\beta_2 + V_4\beta_4)$$

R_1, R_2, ... V_2, V_4 are defined in Appendix B.

Typical numerical values derived from Equation (3.16) are listed in Table 19. Such values, as could be independently determined by exact process compare most favourably with the latter results, Figure 13.

If the function given by Equation (3.15) is used for the case of biaxial compression, with m = 1, the Rayleigh's Quotient is given by

$$\mu = \frac{R_2 R_3 k_m^4 + 2p^2 k_m^2 \gamma^2 R_5 R_6 + p^4 \gamma^4 R_1 R_4 + S}{\gamma^2 k_m^2 R_2 R_9 + p^2 \gamma^4 R_o R_1} \qquad (3.17)$$

where

$m = 1$,

R_o is defined in Appendix B.

The coefficients given by Equation (3.17) are given in Table 20, and are displayed in Figure 9. Good agreement over the whole range of (a/b) ratio and restraint parameters is obtained. Even when a/b is large and $\beta \rightarrow \infty$, the maximum discrepancy is within 2%.

In the next chapter, the exact solution for the buckling of elastically restrained rectangular plates is developed.

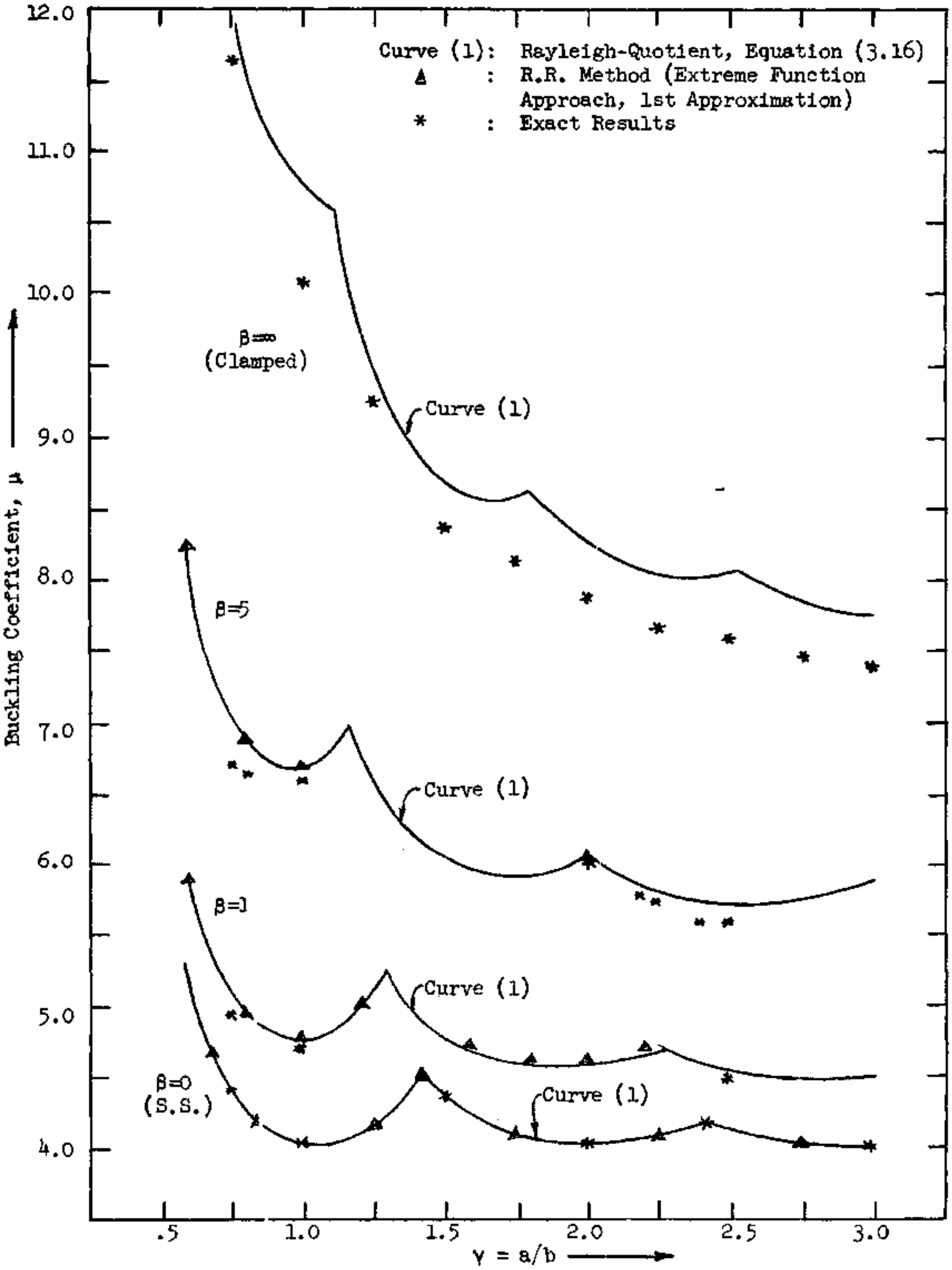
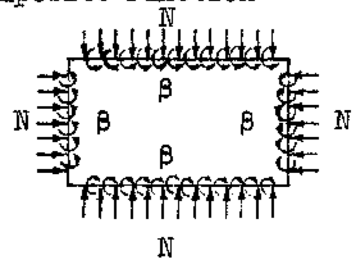


Figure 13. Uniaxially Compressed Rotationally Restrained Rectangular Plate. Buckling Coefficients by Various Methods.

Table 20. Biaxially Compressed Rectangular Plate. Buckling Coefficients by R.R. Process Using Composite Function of True Beam Vibration Shapes.



γ	μ (Rayleigh's Quotient)					
	β = 0	β = 1	β = 5	β = 10	β = 100	β = 10^5
0.6	3.7778	4.3528	6.1003	7.4173	10.6679	11.3513
0.8	2.5625	2.9940	4.1803	4.9849	6.7314	7.0630
1.0	2.0000	2.3693	3.3347	3.9498	5.1650	5.3750
1.2	1.6945	2.0366	2.9117	3.4532	4.4689	4.6337
1.4	1.5102	1.8413	2.6819	3.1963	4.1399	4.2882
1.6	1.3906	1.7184	2.5491	3.0561	3.9794	4.1227
1.8	1.3086	1.6366	2.4686	2.9766	3.9010	4.0439
2.0	1.2500	1.5799	2.4178	2.9302	3.8641	4.0084
2.2	1.2066	1.5391	2.3846	2.9027	3.8488	3.9950
2.4	1.1736	1.5088	2.3624	2.8862	3.8447	3.9930
2.6	1.1479	1.4858	2.3471	2.8764	3.8466	3.9968
2.8	1.1276	1.4679	2.3364	2.8707	3.8514	4.0084
3.0	1.1111	1.4537	2.3287	2.8675	3.8576	4.0112
3.2	1.0977	1.4423	2.3231	2.8660	3.8643	4.0192
3.4	1.0865	1.4331	2.3291	2.8654	3.8709	4.0270
3.6	1.0772	1.4254	2.3160	2.8655	3.8773	4.0345
3.8	1.0693	1.4190	2.3138	2.8660	3.8834	4.0414
4.0	1.0625	1.4136	2.3120	2.8667	3.8890	4.0477

CHAPTER IV

EXACT SOLUTION FOR INSTABILITY LOADS OF AXIALLY COMPRESSED RECTANGULAR PLATE WITH ELASTIC RESTRAINTS ON THE BOUNDARY

The exact solution for the buckling of uniaxially compressed rectangular plate with clamped edges has been obtained by Levy [26]. In this chapter, the exact solution is developed for elastically restrained rectangular plates under uniaxial (Figure 14) as well as biaxial compression.

Consider a simply supported rectangular plate loaded by a concentrated lateral force Q, at a point (x_1,y_1) and axially compressed by uniformly distributed loads along the edges $x = o$ and $x = a$. The system is shown diagramatically in Figure 15.

The Total Potential Energy of the system is given by

$$U = \frac{1}{2}\int_o^a \int_o^b \{(w_{xx} + w_{yy})^2 - 2(1-\nu)(w_{xx}w_{yy} - w_{xy}^2)\}dxdy$$

$$+ \frac{1}{2} N \int_o^a \int_o^b (w_x)^2 \, dxdy - Q\, w(x_1,y_1) \qquad (4.1)$$

where the subscripts denote differentiation with respect to particular variable.

Let w be given by

$$w = a_{mn}\, \varphi(x,y) \qquad (4.2)$$

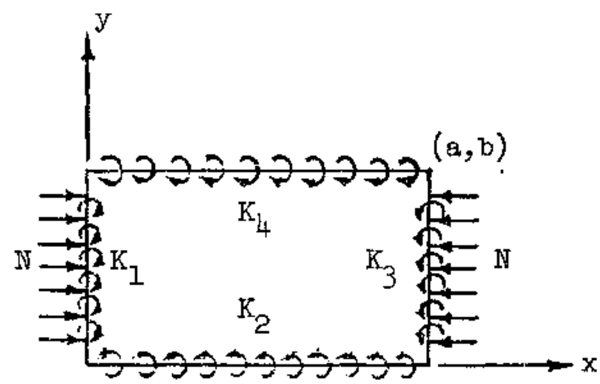


Figure 14. Uniaxially Compressed Rectangular Plate with Unequal Boundary Restraints.

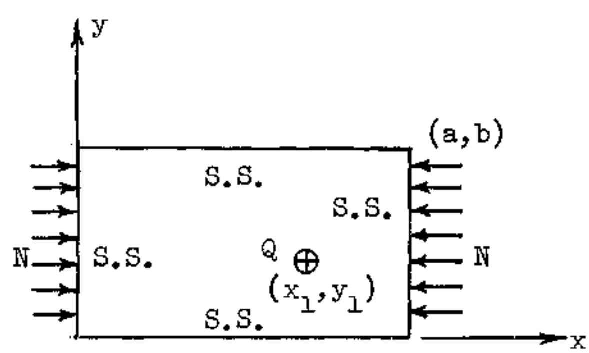


Figure 15. Simply Supported Rectangular Plate Under Uniaxial Load and a Lateral Force.

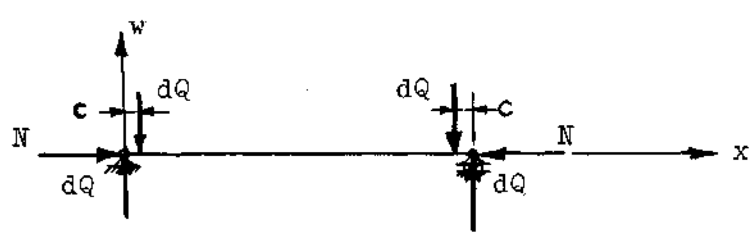


Figure 16. Edge Moment Equivalence.

where m,n represent the number of half waves along the x and y directions respectively.

For equilibrium, $\delta U \equiv 0$.

From this,

the Eulers Equation,

$$\int_0^a \int_0^b (D\nabla^4\varphi + N\varphi_{xx})a_{mn}dxdy - Q\varphi(x_1,y_1) = 0 \tag{4.3}$$

and the boundary conditions,

$$w = o \text{ at } x = o,a \text{ and at } y = o,b; \tag{4.4a}$$

$$D\,w_{xx} + K_3\,w_x = o \quad \text{at } x = a \tag{4.4b}$$

$$D\,w_{xx} - K_1\,w_x = o \quad \text{at } x = o \tag{4.4c}$$

$$D\,w_{yy} + K_4\,w_y = o \quad \text{at } y = b \tag{4.4d}$$

$$D\,w_{yy} - K_2\,w_y = o \quad \text{at } y = o \tag{4.4e}$$

where

$$\nabla^4\varphi = \varphi_{xxxx} + 2\varphi_{xxyy} + \varphi_{yyyy}$$

are obtained.

A rectangular plate with rotational restraints on its boundary is equivalent to the simply supported plate with distributed moments along its edges. Hence the deflection function, w, is taken to be,

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{4.5a}$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \varphi(x,y) \tag{4.5b}$$

Substituting this deflection function in equation (4.3), we obtain

$$a_{mn} = \frac{4Q \sin \frac{m\pi x_1}{a} \sin \frac{n\pi y_1}{b}}{\frac{abD\pi^4}{b^4} \cdot \theta_{mn}} \tag{4.6}$$

where

$$\theta_{mn} = \left\{ \left(\frac{m^2}{\gamma^2} + n^2\right)^2 - \mu \frac{m^2}{\gamma^2} \right\}$$

and

$$\mu = Nb^2/\pi^2 D$$

The applied edge moments along the boundaries are defined as

$$m_{1}(x) = \frac{b^2}{4\pi} \sum_{m=1,2,3} k_{1m} \sin \frac{m\pi x}{a} \text{ at } y = b \tag{4.7a}$$

$$m_{o}(x) = \frac{b^2}{4\pi} \sum_{m=1,2,3} k_{om} \sin \frac{m\pi x}{a} \text{ at } y = o \tag{4.7b}$$

$$m_{1(y)} = \frac{a^2}{4\pi} \sum_{m = 1,3,5} t_{1n} \sin \frac{n\pi y}{b} \text{ at } x = a \tag{4.7c}$$

and

$$m_o(y) = \frac{a^2}{4\pi} \sum_{m = 1,3,5} t_{on} \sin \frac{n\pi y}{b} \text{ at } x = o \tag{4.7d}$$

The moment applied on an edge is equivalent to two equal and opposite lateral forces at the edge separated by a distance c such that $c \to o$, (see Figure 16)

i.e.

$$\text{at } x = 0, \quad m_{o(y)}\, dy = dQ\ c \tag{4.8a}$$

where dQ is applied at the point (c, y_1) and at

$$x = a, \; m_{1(y)}\, dy = dQ\ c \text{ for m odd,}$$

$$- m_{1(y)}\, dy = dQ\ c \text{ for m even,} \tag{4.8b}$$

where

dQ is applied at the point $(a-c, y_1)$

In y direction, however, only odd number of half waves are considered since only one half wave yields minimum critical load. Hence, we have

$$\text{at } y = 0, \quad m_{o(x)}\, dx = dQ\ c \tag{4.8c}$$

where dQ is applied at the point $(x_{1,c})$

and

$$\text{at } y = b, \qquad m_{o(x)}\, dx = dQ\, C \tag{4.8d}$$

where dQ is applied at the point $(x_1,\ b\text{-}c)$.

Employing equations (4.7) and (4.8) in (4.6), integrating around the boundaries of the plate and taking the limit as $c \to o$, we obtain,

$$a_{mn} = \frac{m(t_{1n} + t_{on}) + n(k_{1m} + k_{om})}{\frac{D\pi^4}{b^4} \cdot 2\ \theta_m} \tag{4.9}$$

and

$$w = \sum_{\substack{m = 1,3,5 \\ \text{or}\ \ m = 2,4,6}}^{\infty} \ \sum_{n = 1,3,5}^{\infty} \frac{\{m(t_{1n}+t_{on})+n(k_{1m}+k_{om})\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{\frac{D\pi^4}{b^4} \cdot 2\ \theta_{mn}} \tag{4.10}$$

The boundary condition at $x = a$, from Equation (4.4b) is given by

$$w_{xx} + \frac{\beta_3}{b}\, w_x = 0 \tag{4.11}$$

The moment $m_{1(y)}$ along the edge $x = a$ is also given by

$$\begin{aligned} m_{1(y)} &= -\ D(w_{xx} + \nu\, w_{yy}) \\ &= -\ D\, w_{xx} \end{aligned} \tag{4.12}$$

Hence, the boundary conditions (4.4a) through (4.4d) become

$$m_{1(y)} - \frac{\beta_3}{b} w_x = 0 \qquad \text{at } x = a \tag{4.13a}$$

$$m_o(y) + \frac{\beta_1}{b} w_x = 0 \qquad \text{at } x = 0 \tag{4.13b}$$

$$m_{1(x)} - \beta_4 w_y = 0 \qquad \text{at } y = b \tag{4.13c}$$

and

$$m_{o(x)} + \beta_2 w_y = 0 \qquad \text{at } y = 0 \ . \tag{4.13d}$$

Equations (4.7), (4.10) and (4.13) together give,

$$\frac{\pi^2}{4} k_{1m} + \beta_4 k_m R_m + \beta_4 \sum_{n=1,3,5}^{\infty} \frac{t_n \, m \cdot n}{\theta_{mn}} = 0 \qquad \begin{array}{l}\text{for } m = 1,3,5\cdots \\ \text{or } m = 2,4,6\cdots\end{array} \tag{4.14a}$$

$$\frac{\pi^2}{4} k_{om} + \beta_2 k_m R_m + \beta_2 \sum_{n=1,3,5}^{\infty} \frac{t_n \, m \cdot n}{\theta_{mn}} = 0 \qquad \begin{array}{l}\text{for } m = 1,3,5\cdots \\ \text{or } m = 2,4,6\cdots\end{array} \tag{4.14b}$$

$$\gamma^3 \frac{\pi^2}{4} t_{1n} + \beta_3 \sum_{\substack{m=1,3,5 \\ \text{or } m=2,4,6}}^{\infty} \frac{k_m mn}{\theta_{mn}} + \beta_3 t_n (R_{no} \text{ or } R_{ne}) = 0$$

$$\text{for } n = 1,3,5\cdots \tag{4.14c}$$

$$\gamma^3 \frac{\pi^2}{4} t_{on} + \beta_1 \sum_{\substack{m=1,3,5 \\ \text{or } m=2,4,6}}^{\infty} \frac{k_m mn}{\theta_{mn}} + \beta_1 t_n (R_{no} \text{ or } R_{ne}) = 0$$

$$\text{for } n = 1,3,5 \tag{4.14d}$$

where

$$k_m = \frac{1}{2}(k_{1m} + k_{om})$$

$$t_n = \frac{1}{2}(t_{1n} + t_{on})$$

$$R_m = \sum_{n=1,3,5}^{\infty} \frac{n^2}{\theta_{mn}}$$

$$R_{n_o} = \sum_{m=1,3,5}^{\infty} \frac{m^2}{\theta_{mn}}$$

$$R_{n_e} = \sum_{m=2,4,6}^{\infty} \frac{m^2}{\theta_{mn}}$$

The characteristic determinant associated with the constants t_{01}, t_{11}, k_{01}, k_{11}, k_{03} and k_{13} is given, for example, by

$$\Delta = \begin{vmatrix} \frac{\gamma^3\pi^2}{2} + \beta_1 R_{10} & \beta_1 R_{10} & \beta_1 \frac{1}{\theta_{11}} & \beta_1 \frac{1}{\theta_{11}} & \beta_1 \frac{3}{\theta_{31}} & \beta_1 \frac{3}{\theta_{31}} \\ \beta_3 R_{10} & \frac{\gamma^3\pi^2}{2} + \beta_3 R_{10} & \beta_3 \frac{1}{\theta_{11}} & \beta_3 \frac{1}{\theta_{11}} & \beta_3 \frac{3}{\theta_{31}} & \beta_3 \frac{3}{\theta_{31}} \\ \beta_2 \frac{1}{\theta_{11}} & \beta_2 \frac{1}{\theta_{11}} & \frac{\pi^2}{2} + \beta_2 R_1 & \beta_2 R_1 & 0 & 0 \\ \beta_4 \frac{1}{\theta_{11}} & \beta_4 \frac{1}{\theta_{11}} & \beta_4 R_1 & \frac{\pi^2}{2} + \beta_4 R_1 & 0 & 0 \\ \beta_2 \frac{3}{\theta_{31}} & \beta_2 \frac{3}{\theta_{31}} & 0 & 0 & \frac{\pi^2}{2} + \beta_2 R_3 & \beta_2 R_3 \\ \beta_4 \frac{3}{\theta_{31}} & \beta_4 \frac{3}{\theta_{31}} & 0 & 0 & \beta_4 R_3 & \frac{\pi^2}{2} + \beta_4 R_3 \end{vmatrix} \quad (4.15)$$

The summations given in reference [26] are used in the evaluation of the above characteristic determinant. They are defined as follows:

$$R_m = \sum_{n=1,3,5}^{\infty} \frac{n^2}{\theta_{mn}}$$

$$= \sum_{n=1,3,5}^{\infty} \frac{n^2}{\left\{\left(\frac{m^2}{\gamma^2} + n^2\right)^2 - \mu \frac{m^2}{\gamma^2}\right\}}$$

$$= \frac{\gamma}{4m\sqrt{\mu}} \left\{\frac{\pi}{2} \sqrt{\frac{m^2}{\gamma^2} + \frac{m}{\gamma}\sqrt{\mu}} \tanh \frac{\pi}{2} \sqrt{\frac{m^2}{\gamma^2} + \frac{m}{\gamma}\sqrt{\mu}}\right.$$

$$\left. - \frac{\pi}{2} \sqrt{\frac{m^2}{\gamma^2} - \frac{m}{\gamma}\sqrt{\mu}} \tanh \frac{\pi}{2} \sqrt{\frac{m^2}{\gamma^2} - \frac{m}{\gamma}\sqrt{\mu}}\right\} \tag{4.16}$$

$$Rn_o = \sum_{m=1,3,5}^{\infty} \frac{m^2}{\theta_{mn}}$$

$$= \sum_{m=1,3,5}^{\infty} \frac{m^2}{\left\{\left(\frac{m^2}{\gamma^2} + n^2\right)^2 - \mu \frac{m^2}{\gamma^2}\right\}}$$

$$= \frac{\gamma^2}{2\mu\sqrt{1 - \frac{4n^2}{\mu}}} \left\{\frac{\pi\gamma n}{2} \sqrt{1 - \frac{\mu}{2n^2}\left[1 - \sqrt{\left(1 - \frac{4n^2}{\mu}\right)}\right]}\right. .$$

$$\tanh \frac{\pi\gamma n}{2} \sqrt{1 - \frac{\mu}{2n^2}\left[1 - \sqrt{\left(1 - \frac{4n^2}{\mu}\right)}\right]}$$

$$- \frac{\pi\gamma n}{2}\sqrt{1 - \frac{\mu}{2n^2}\left[1 + \sqrt{1 - \frac{4n^2}{\mu}}\right]}\tanh \frac{\pi\gamma n}{2}\sqrt{1 - \frac{\mu}{2n^2}\left[1 + \sqrt{1 - \frac{4n^2}{\mu}}\right]}\Bigg\}$$

(4.17)

$$R_{n_e} = \sum_{m=2,4,6}^{\infty} \frac{m^2}{\left\{\left(\frac{m^2}{\gamma^2} + n^2\right)^2 - \mu \frac{m^2}{\gamma^2}\right\}}$$

$$= \frac{\gamma^2}{2\mu \sqrt{1 - \frac{4n^2}{\mu}}} \left\{\frac{\pi\gamma n}{2}\sqrt{1 - \frac{\mu}{2n^2}\left[1 - \sqrt{\left(1 - \frac{4n^2}{\mu}\right)}\right]}\right.$$

$$\coth \frac{\pi\gamma n}{2}\sqrt{1 - \frac{\mu}{2n^2}\left[1 - \sqrt{1 - \frac{4n^2}{\mu}}\right]}$$

$$- \frac{\pi\gamma n}{2}\sqrt{1 - \frac{\mu}{2n^2}\left[1 + \sqrt{1 - \frac{4n^2}{\mu}}\right]} \coth \frac{\pi\gamma n}{2}\sqrt{1 - \frac{\mu}{2n^2}\left[1 + \sqrt{1 - \frac{4n^2}{\mu}}\right]}\Bigg\}$$

(4.18)

For biaxial compression the analysis is made in a similar fashion. For this case the characteristic determinant is given by Equation (4.15), as before.

But now,

$$\theta_{mn} = \left\{ \left(\frac{m^2}{\gamma^2} + n^2\right)^2 - \mu\left(\frac{m^2}{\gamma^2} + n^2\right) \right\}$$

and the summations R_m and R_{n_o} are developed as follows.

$$R_m = \sum_{n=1,3,5}^{\infty} \frac{n^2}{\theta_{mn}}$$

$$= \sum_{n=1,3,5}^{\infty} \frac{n^2}{\left\{\left(\frac{m^2}{\gamma^2} + n^2\right)^2 - \mu\left(\frac{m^2}{\gamma^2} + n^2\right)\right\}}$$

$$= \sum_{n=1,3,5}^{\infty} \frac{n^2}{\left(\frac{m^2}{\gamma^2} + n^2\right)\left(\frac{m^2}{\gamma^2} + n^2 - \mu\right)}$$

$$= \frac{m^2}{\gamma^2\mu} \sum_{n=1,3,5}^{\infty} \frac{1}{\left(\frac{m^2}{\gamma^2} + n^2\right)} + \frac{\mu - \frac{m^2}{\gamma^2}}{\mu} \sum_{n=1,3,5}^{\infty} \frac{1}{\left(\frac{m^2}{\gamma^2} + n^2 - \mu\right)}$$

$$= \frac{m\pi}{4\gamma\mu} \left\{\coth\left(\frac{m\pi}{\gamma}\right) - \operatorname{cosech}\left(\frac{m\pi}{\gamma}\right)\right\} - \frac{\pi\sqrt{\frac{m^2}{\gamma^2} - \mu}}{4\mu}$$

$$\left\{\coth\pi \sqrt{\frac{m^2}{\gamma^2} - \mu} - \operatorname{cosech} \pi \sqrt{\frac{m^2}{\gamma^2} - \mu}\right\} \qquad (4.19)$$

$$R_{n_o} = \sum_{m=1,3,5}^{\infty} \frac{m^2}{\theta_{mn}} = \sum_{m=1,3,5}^{\infty} \frac{m^2}{\left\{\left(\frac{m^2}{\gamma^2} + n^2\right)^2 - \mu\left(\frac{m^2}{\gamma^2} + n^2\right)\right\}}$$

$$= \frac{n^2\gamma^2}{\mu} \sum_{m=1,3,5}^{\infty} \frac{1}{\left(\frac{m^2}{\gamma^2} + n^2\right)} + \frac{(\mu-n^2)\gamma^2}{\mu} \sum_{m=1,3,5}^{\infty} \frac{1}{\left(\frac{m^2}{\gamma^2} + n^2 - \mu\right)}$$

$$= \frac{n\pi\gamma^3}{4\mu} \{\coth(n\pi\gamma) - \operatorname{cosech}(n\pi\gamma)\}$$

$$- \frac{\pi\sqrt{n^2-\mu}\cdot\gamma^3}{4\mu} \{\coth \pi \gamma\sqrt{n^2-\mu} - \operatorname{cosech} \pi\gamma\sqrt{n^2-\mu}\} \quad (4.20)$$

It should be noted that if all the edges of the plate have equal restraints, then $k_{om} = k_{1m}$ and $t_{on} = t_{1n}$, and hence equations (4.14) reduce to

$$\pi^2\gamma^3 t_{on} + \beta \sum_{\substack{m=1,3,5 \\ \text{or } m=2,4,6}}^{\infty} \frac{m\ n\ k_{om}}{\theta_{mn}} + \beta(Rn_o \text{ or } Rn_e) = 0 \ , \qquad n=1,3,5\cdots \quad (4.21a)$$

and

$$\frac{\pi^2}{4} k_{om} + \beta\ k_{om}R_m + \beta \sum_{n=1,3,5}^{\infty} \frac{t_{on}\ mn}{\theta_{mn}} = 0 \ , \quad m=1,3,5\cdots \text{ or } m=2,4,6\cdots \quad (4.21b)$$

The characteristic determinant corresponding to the coefficients t_{01}, k_{01} and k_{03} is given by,

$$\Delta = \begin{vmatrix} \frac{\gamma^3\pi^2}{4} + \beta R_{10} & \beta\frac{1}{\theta_{11}} & \beta\frac{3}{\theta_{31}} \\ \beta\frac{1}{\theta_{11}} & \frac{\pi^2}{4} + \beta R_1 & 0 \\ \beta\frac{3}{\theta_{31}} & 0 & \frac{\pi^2}{4} + \beta R_3 \end{vmatrix} \quad (4.22)$$

Buckling coefficients have been computed for various aspect ratios and restraint parameters. These are given in Table 19 for the uniaxial compression case and Table 21 for the biaxial. The variation in the value of the determinant for a given value of aspect ratio ($\gamma = 2$) and a prescribed value of the boundary restraint parameter ($\beta = 10^6$), as the compressive force (N) changes, is portrayed in Figure 17. A similar situation, for $\gamma = 3.5$ and $\beta = 5$, is shown in Figure 18. It is seen from these graphs that the values of the determinant fluctuate between $\pm \infty$. Consequently there are singularities. Thus to obtain the critical load coefficient which corresponds to the zero value of the determinant, we must exercise a great care.

A finite difference approach is outlined in Appendix C. A comparison of the buckling coefficients for a biaxially compressed square plate for edge restraint parameters 5 and 10, as derived from R.R., finite difference and exact solution, is presented in Table 22. As expected, the R.R. values are higher and the finite difference values are lower than the exact values. The discrepancies, however, are insignificant.

A flow chart for the iteration process used for the numerical solution of various characteristic equations in the thesis is given in Appendix D.

Table 21. Biaxially Compressed Rectangular Plate with Equal Boundary Restraints. Buckling Coefficients by Rayleigh-Ritz Method (Table 14) and Exact Solutions.

β \ γ	1.0	1.5	2.0	2.5	3.0	3.5
1	2.3692*					
	(2.3704)**					
5	3.3285	2.6500	2.4150	2.3500	2.3000	
	(3.3456)		(2.4179)		(2.3280)	
10	3.9256	3.1100	2.9000	2.8450	2.8400	
	(3.9617)		(2.9174)		(2.8513)	
10000	5.2400	4.1100	3.9200	3.9000		3.8850
	(5.3113)		(3.9287)			

* EXACT

** R.R.

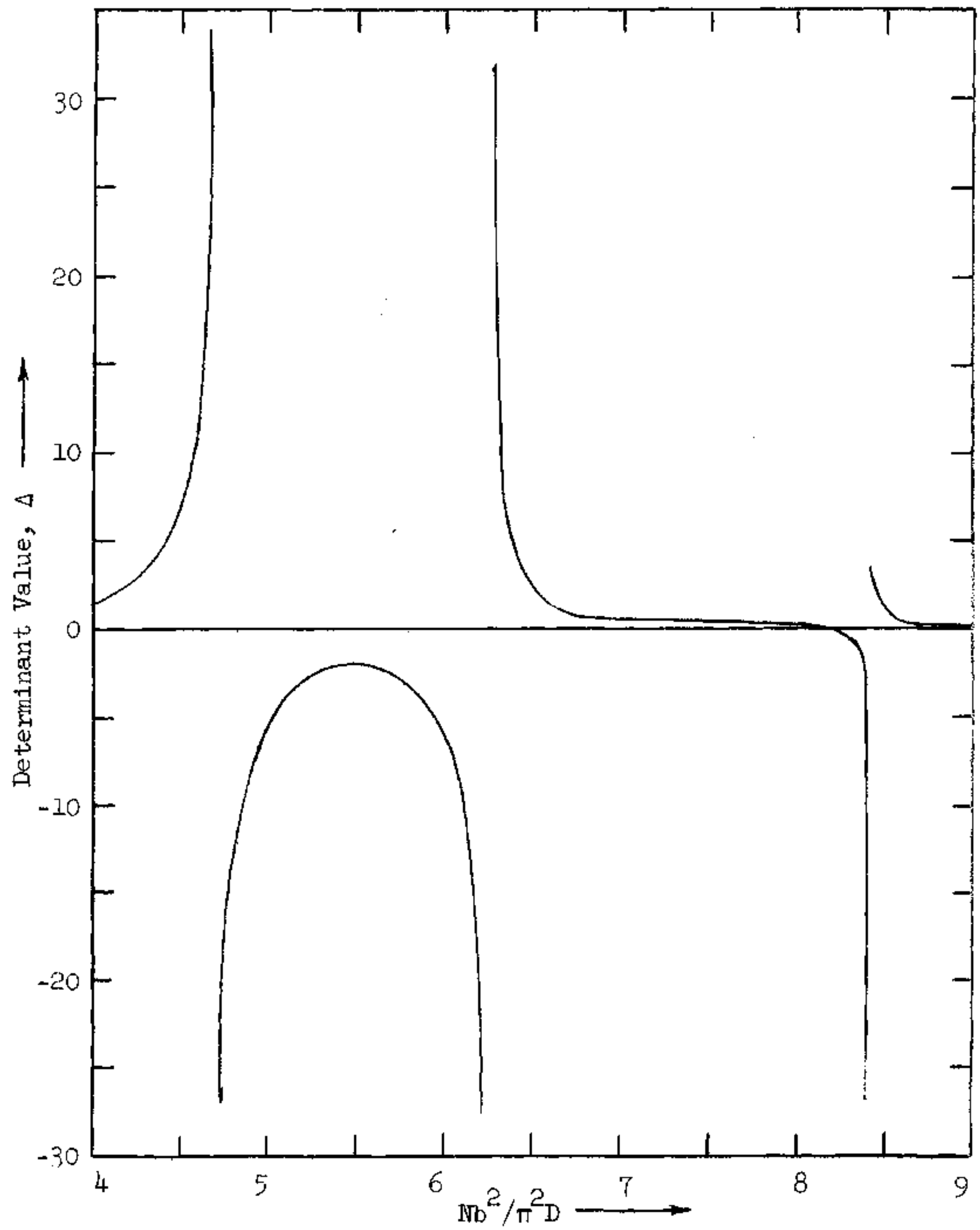


Figure 17. Biaxially Compressed Rectangular Plate with Equal Rotational Restraints on the Boundaries. Value of the Characteristic Determinant VS Axial Load, for $\gamma = 2$, $\beta = 10^6$.

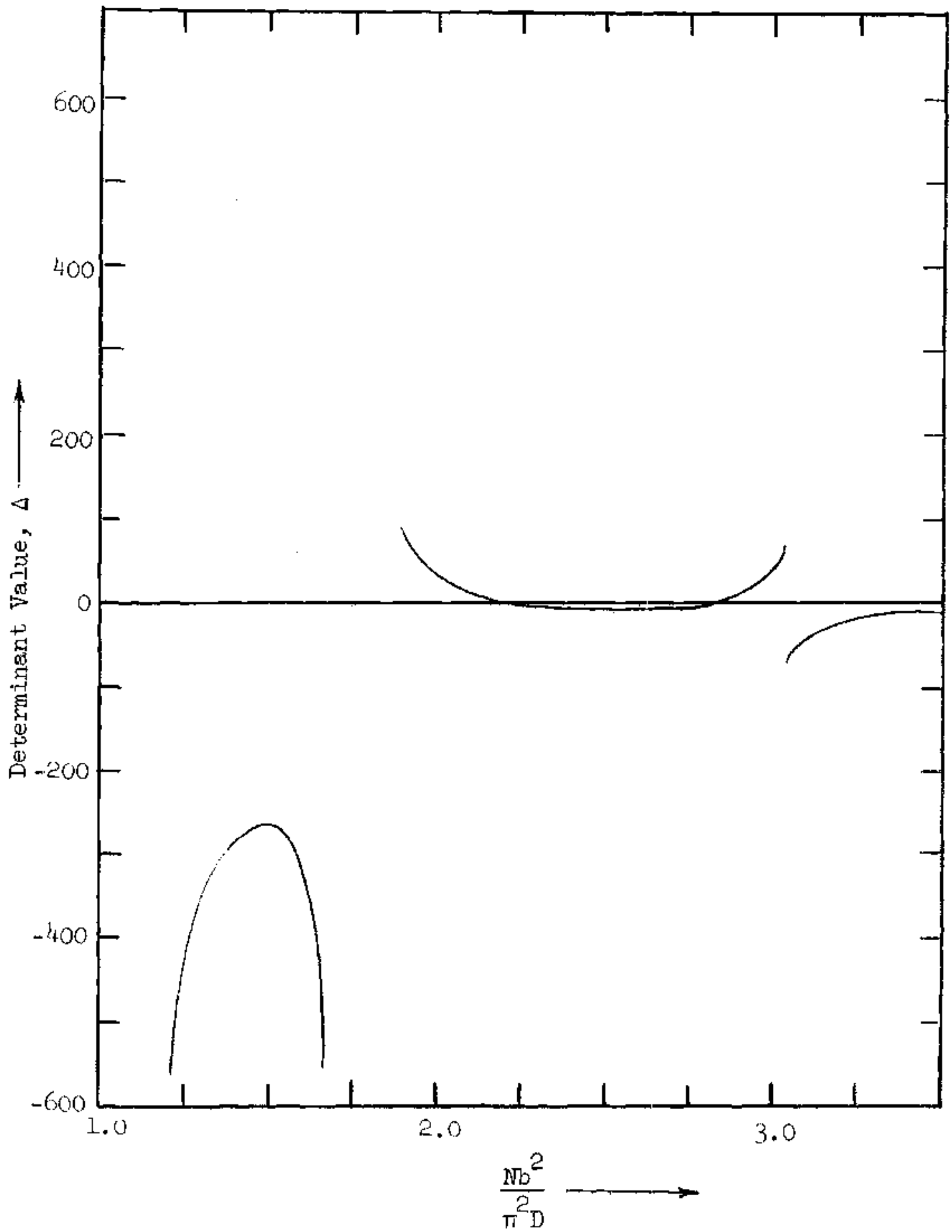


Figure 18. Biaxially Compressed Rectangular Plate with Equal Rotational Restraints on the Boundaries. Value of the Characteristic Determinant VS Axial Load for $\gamma = 3.5$, $\beta = 5$.

Table 22. Buckling Coefficients (μ) Obtained by Various Methods for Biaxially Compressed Identically Restrained Square Plate.

Restraint Parameter	Rayleigh-Ritz	Exact Solution	Finite Difference
5	3.34563	3.32847	3.22078
10	3.96173	3.92562	3.77309

CHAPTER V

CONCLUSION

The choice of a suitable displacement function is the key to the successful use of the Rayleigh-Ritz method for determining critical loads for elastically restrained columns and plates. A deficit in the current literature on the subject existed viz. no systematic procedure for making an appropriate choice had been defined. The studies outlined in this thesis contribute much to the resolution of this deficiency.

Firstly, a linear combination of the functions corresponding to the independent combinations of the extreme values of the elastic restraints, when used with R.R. method, gives eigenvalues well within normal engineering accuracy.

Secondly, a power series of the product of these extreme functions when employed with R.R. process, ensures outstanding convergence.

Also, using the linear terms in the assume sequence of functions and satisfying the restraint boundary conditions, very simple and convenient expressions for eigenvalues can be derived.

Using the techniques outlined in this thesis, much more complex problems that are not normally tractable, can be readily treated.

APPENDIX A

RAYLEIGH RITZ METHOD (VARIATIONAL METHOD)*

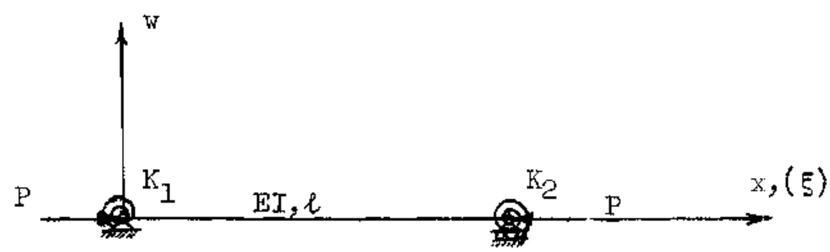


Figure A.1

Consider, for example, a column with end rotational restraints K_1, K_2. The Total Potential Energy functional for this case is given by

$$U = \frac{EI}{2\ell^3} \int_o^1 (w_{\xi\xi})^2 \, d\xi - \frac{P}{2\ell} \int_o^1 (w_\xi)^2 \, d\xi$$

$$+ \frac{1}{2} \frac{K_1}{\ell^2} [w_\xi(o)]^2 + \frac{1}{2} \frac{K_2}{\ell^2} [w_\xi(1)]^2 \qquad (A.1)$$

where subscript denotes differentiation. We seek those w^s, which make U, in equation (A.1), stationary-truly a minimum.

Let us assume that

---

*A brief summary is presented since it is available in numerous text books.

$$w(\xi) = \varphi(\xi) + \epsilon\, \eta(\xi) \tag{A.2}$$

where

$\varphi(\xi)$ are admissible functions satisfying the prescribed geometric end conditions, are continuous and have continuous derivatives,

ϵ is a small parameter and

$\eta(\xi)$ are arbitrary, continuous and have continuous derivatives and satisfy the prescribed geometric end conditions. Substituting equation (A.2) in (A.1), we obtain $U(\epsilon)$ and a necessary condition for an extremum of this function is

$$\lim_{\epsilon \to 0} \frac{\partial U}{\partial \epsilon} = 0 \tag{A.3}$$

This condition gives, the Eulers Equation:

$$EI\, w_{\xi\xi\xi\xi} + \ell^2 P w_{\xi\xi} = 0 \tag{A.4}$$

and Natural Boundary Conditions:

$$EI\, w_{\xi\xi\xi} + \ell^2 P w_{\xi} = 0 \quad \text{at} \quad \xi = 0 \text{ and } 1, \tag{A.5a}$$

$$EI\, w_{\xi\xi} + K_2 \ell w_{\xi} = 0 \quad \text{at} \quad \xi = 1 \tag{A.5b}$$

$$EI\, w_{\xi\xi} - K_1 \ell w_{\xi} = 0 \quad \text{at} \quad \xi = 0 \tag{A.5c}$$

To the above conditions, the geometric boundary conditions also must be added

i.e.

$$w = 0 \quad \text{and} \quad w_\xi = 0 \quad \text{at} \quad \xi = 0 \text{ and } 1 \tag{A.6}$$

In Rayleigh-Ritz process, we select approximate w of the form,

$$w = \sum_{i=1}^{n} a_i \varphi_i \tag{A.7}$$

where

a_i are undetermined constants,

φ_i are independent admissible functions such that each φ_i satisfies at least the essential boundary conditions (sometimes referred to as 'geometric' or 'rigid' boundary conditions in the literature).

It can be noted that for elastically restrained structures, an essential condition in the extreme case may not be so when finite nonzero restraints are present and only the pertinent extreme function need satisfy these. The assumed function (A.7) is substituted in the total potential energy expression (A.1) and we obtain a function $U(a_i, \lambda)$.

For equilibrium, we get

$$\frac{\partial U}{\partial a_i} = 0 , \qquad i = 1,2,3,\cdots n \tag{A.8}$$

For a nontrivial solution of the resulting n homogeneous simultaneous equations, the determinant of the coefficients, a_i, must vanish. The numerical solution of this system of equations gives the associated eigenvalues.

Rayleigh-Quotient is the expression which results from the R.R. process when only one arbitrary constant is involved.

APPENDIX B

UNIAXIALLY LOADED RECTANGULAR PLATE WITH EQUAL UNIFORM RESTRAINTS ON ALL EDGES

The approximation function is given by equation (3-14), i.e.

$$w = a_1 \cos n\pi\xi \cos \pi\eta$$

$$+ a_2(C_m \cosh k_m\xi - c_{mh} \mathrm{Cos}\, k_m\xi)(D \cosh p\eta - D_h \cos p\eta) \quad \text{(B-1)}$$

where relevant quantities are defined in equation (3-14). Using this function with Rayleigh-Ritz process we obtain for nontrivial solution

$$\alpha^2(A_2A_6 - A_4^2) - \alpha(A_2A_5 + A_1A_6 - 2A_3A_4) + A_1A_5 - A_3^2 = 0 \quad \text{(B-2)}$$

where

$$A_1 = \frac{\pi^4}{4}(n^2 + \gamma^2)^2 + \beta\pi^2(\gamma + \gamma^4)$$

$$A_2 = \gamma^2 \frac{\pi^4}{4}$$

$$A_3 = (S_4M_5\pi^2k_m^2 + \pi^2p^2\gamma^2M_4S_5)(1 + \gamma^2)$$

$$A_4 = \pi^3\gamma^2k_mM_5(C_mI_8 + I_{10}C_{mh})$$

$$A_5 = k_m^4S_1M_2 + 2\gamma^2k_m^2p^2S_3M_3 + \gamma^4p^4M_1S_2$$

$$A_6 = \pi^2\gamma^2k_m^2M_2(C_m^2I_2 + 2C_mC_{mh}I_5 + C_{mh}^2I_4)$$

$$S_1 = C_m^2 I_1 + 2\ C_m C_{mh}\ I_9 + C_{mh}^{\ 2}\ I_3$$

$$M_1 = D^2\ J_1 + 2\ D\ D_h\ J_9 + D_h^2\ J_3$$

$$S_2 = C_m^2\ I_1 - 2\ C_m C_{mh}\ I_9 + C_{mh}^{\ 2}\ I_3$$

$$M_2 = D^2\ J_1 - 2\ D\ D_h\ J_9 + D_h^2\ J_3$$

$$S_3 = C_m^2\ I_1 - C_{mh}^{\ 2}\ I_3$$

$$M_3 = D^2\ J_1 - D_h^2\ J_3$$

$$S_4 = C_m\ I_6 + C_{mh}\ I_7$$

$$M_4 = D\ J_6 + D_h\ J_7$$

$$S_5 = C_m\ I_6 - C_{mh}\ I_7$$

$$M_5 = D\ J_6 - D_h\ J_7$$

Integrals I_1, I_2, ... I_{10}, J_1, J_2, ... are given in Table B-1. Terms R_1, R_2, ... V_1, V_2, ... used in equations (3-16) and (3-17) are defined as follows.

$$R_1 = C_1^2 I_1 + C_2^2 I_2 + C_3^2 I_3 + C_4^2 I_4 + 2C_1 C_3 I_9 + 2C_2 C_4 I_5$$

$$R_2 = D_1^2 J_1 + D_2^2 J_2 + D_3^2 J_3 + D_4^2 J_4 + 2D_1 D_3 J_9 + 2D_2 D_4 J_5$$

$$R_3 = R_1 - 4C_1 C_3\ I_9 - 4C_2 C_4\ I_5$$

$$R_4 = R_2 - 4\, D_1 D_3\, J_9 - 4\, D_2 D_4\, J_5$$

$$R_5 = C_1^2\, I_1 + C_2^2 I_2 - C_3^2 I_3 - C_4^2 I_4$$

$$R_6 = D_1^2 J_1 + D_2^2 J_2 - D_3^2 J_3 - D_4^2 J_4$$

$$R_9 = C_1^2 I_2 + C_2^2 I_1 + C_3^2 I_4 + C_4^2 I_3 - 2C_1 C_3 I_5 + 2C_2 C_4 I_9$$

$$R_0 = D_1^2 J_2 + D_2^2 J_1 + D_3^2 J_4 + D_4^2 J_3 - 2D_1 D_3 J_5 - 2D_2 D_4 J_9$$

$$V_1 = - C_1 \sinh \frac{k_m}{2} + C_2 \cosh \frac{k_m}{2} + C_3 \sin \frac{k_m}{2} + C_4 \cos \frac{k_m}{2}$$

$$V_2 = - D_1 \sinh \frac{p}{2} + D_2 \cosh \frac{p}{2} + D_3 \sin \frac{p}{2} + D_4 \cos \frac{p}{2}$$

$$V_3 = C_1 \sinh \frac{k_m}{2} + C_2 \cosh \frac{k_m}{2} - C_3 \sin \frac{k_m}{2} + C_4 \cos \frac{k_m}{2}$$

$$V_4 = D_1 \sinh \frac{p}{2} + D_2 \cosh \frac{p}{2} - D_3 \sin \frac{p}{2} + D_4 \cos \frac{p}{2}$$

Table B-1. Integrals used in equation (B.2).

Integrals I_i and J_i are defined as follows:

$$I_i = \int_{-\frac{a}{2}}^{\frac{a}{2}} f_i(k_m,\xi)d\xi = F_i(k_m,a)$$

$$J_i = \int_{-\frac{b}{2}}^{\frac{b}{2}} f_i(p,\eta)d\eta = F_i(p,b)$$

i	$f_i(k_m,\xi)$	$F_i(k_m,a) = I_i$
1	$\cosh^2 k_m\xi$	$\frac{a}{2}\left(1 + \frac{\sinh k_m}{k_m}\right)$
2	$\sinh^2 k_m\xi$	$\frac{a}{2}\left(\frac{\sinh k_m}{k_m} - 1\right)$
3	$\cos^2 k_m\xi$	$\frac{a}{2}\left(1 + \frac{\sin k_m}{k_m}\right)$

(Continued)

Table B-1. Integrals used in equation (B.2). (Continued)

i	$f_i(k_m,\xi)$	$F_i(k_m,a) = I_i$
4	$\sin^2 k_m\xi$	$\frac{a}{2}\left(1 - \frac{\sin k_m}{k_m}\right)$
5	$\sin k_m\xi \sinh k_m\xi$	$\frac{a}{k_m}\left(\cos\frac{k_m}{2}\sinh\frac{k_m}{2} - \cosh\frac{k_m}{2}\sin\frac{k_m}{2}\right)$
6	$\cos m\pi\xi \cosh k_m\xi$	$\frac{a}{k_m^2 + m^2\pi^2}\left(2k_m\cos\frac{m\pi}{2}\sinh\frac{k_m}{2} + 2m\pi\sin\frac{m\pi}{2}\cosh\frac{k_m}{2}\right)$
7	$\cos m\pi\xi \cos k_m\xi$	$\frac{a.\sin\left(\frac{m\pi - k_m}{2}\right)}{(m\pi - k_m)} + \frac{a.\sin\left(\frac{m\pi - k_m}{2}\right)}{(m\pi + k_m)}$
8	$\sin m\pi\xi \sinh k_m\xi$	$\frac{2a}{k_m^2 + m^2\pi^2}\left(k_m\sin\frac{m\pi}{2}\cosh\frac{k_m}{2} - m\pi\cos\frac{m\pi}{2}\sinh\frac{k_m}{2}\right)$

(Continued)

Table B-1. Integrals used in equation (B.2). (Continued)

i	$f_i(k_m,\xi)$	$F_i(k_m,a) = I_i$
9	$\cos k_m\xi \cosh k_m\xi$	$\frac{a}{k_m}\left(\cos \frac{k_m}{2} \sinh \frac{k_m}{2} + \cosh \frac{k_m}{2} \sin \frac{k_m}{2}\right)$
10	$\sin m\pi\xi \sin k_m\xi$	$\frac{a \sin\left(\frac{m\pi - k_m}{2}\right)}{(m\pi - k_m)} - \frac{a \sin\left(\frac{m\pi + k_m}{2}\right)}{(m\pi + k_m)}$

APPENDIX C

FINITE DIFFERENCE APPROACH FOR ELASTICALLY RESTRAINED RECTANGULAR PLATES

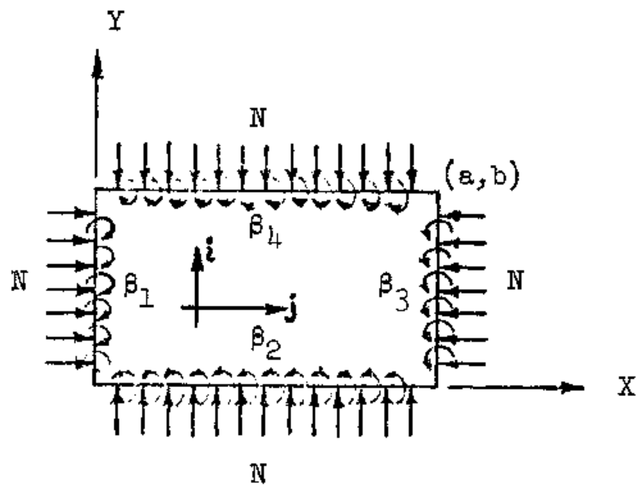


Figure C-1. Elastically Restrained Rectangular Plate Under Biaxial Compression.

The governing differential equation is given by

$$\nabla^4 W + \frac{N}{D} (W_{XX} + W_{YY}) = 0 \qquad (C.1)$$

Nondimensional parameters are introduced as follows.

$$W = wh \ ,$$

$$X = xb \ ,$$

$$Y = yb \ ,$$

$$\alpha = \frac{Nb^2}{D}$$

Governing differential equation then becomes

$$\nabla^4 w + \pi^2 \mu (w_{xx} + w_{yy}) = 0 \tag{C.2}$$

Defining $w_{xx} = \psi$ and $w_{yy} = \zeta$, Equation (C.2) becomes

$$(\psi_{xx} + \psi_{yy} + \zeta_{xx} + \zeta_{yy}) + \pi^2 \mu (\psi + \zeta) = 0 \tag{C.3}$$

Following equations are used along with Equation (C.3)

$$w_{xx} = \psi \tag{C.4}$$

and

$$\psi_{yy} = \zeta_{xx} \tag{C.5}$$

Using central differences, equations (C.3), (C.4) and (C.5) are written at every nodal point (i,j) on the plate and the boundary conditions are used. The resulting characteristic equation is then numerically solved using Potter's method [27].

APPENDIX D

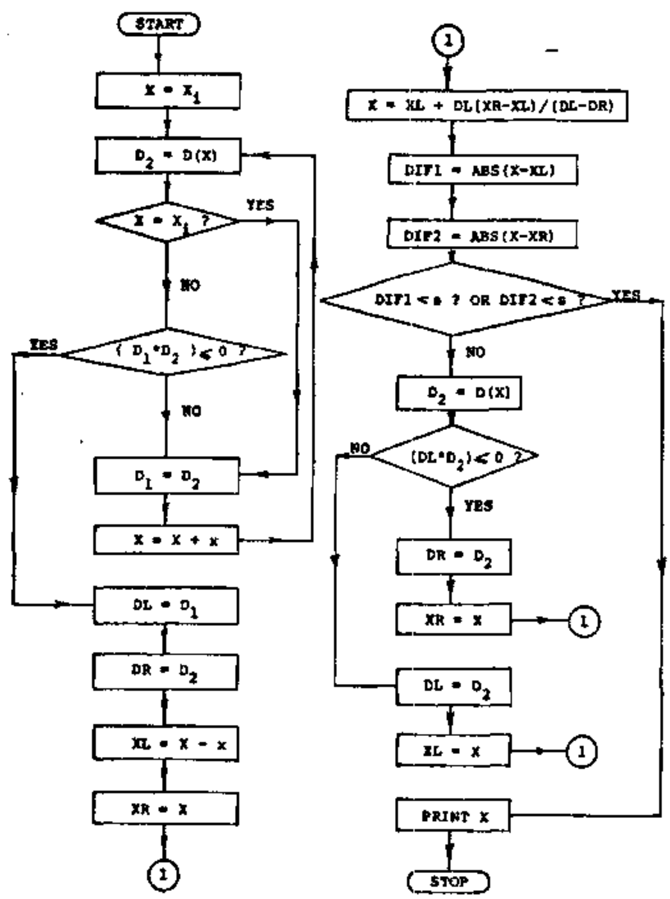
FLOW CHART FOR THE NUMERICAL PROCESS USED IN THE THESIS

The characteristic equation:

$$D(X) = 0$$

X_i and x are the initial value of and increment in the eigenvalue x respectively and s is the desired accuracy of the solution.

FLOW CHART



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VITA

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Sopan Chaudhari has an "American Family", Mr. and Mrs. Donald Kalet, in Atlanta, who give him the affection and constant encouragement, while he is far away from his home.